Instabilities in Explicit Super-Time-Stepping Schemes

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Why STS Schemes?

- stability criteria much more practical than explicit Euler for diffusion equations:

$$s \propto \frac{1}{\sqrt{\Delta t_{\text{explicit}}}}$$

where $\Delta t_{\text{explicit}} \propto \frac{m^2}{n}$ with m number of space-steps, n number of time-steps, s number of stages.

- still explicit: easy to adapt to multi-dimensional problems and non-linear problems

Some STS schemes

Diffusion equation:

$$\frac{\partial f}{\partial t}(x,t) = \mathcal{L}\left(f(x,t), x, t\right) . \tag{1}$$

And the RKC, RKL, RKG schemes read (Verwer 1996, Meyer 2014)

$$\hat{f}^0 = f(t_j)\,, \tag{2a}$$

$$\hat{f}^1 = \hat{f}^0 + \tilde{\lambda}_1 k_j \mathcal{L}\left(\hat{f}^0\right) , \tag{2b}$$

$$\begin{aligned}\hat{f}^\eta = {} & \lambda_\eta \hat{f}^{\eta-1} + \nu_\eta \hat{f}^{\eta-2} + (1-\lambda_\eta-\nu_\eta)\hat{f}^0 \\ & + \tilde{\lambda}_\eta k_j \mathcal{L}\left(\hat{f}^{\eta-1}\right) + \tilde{\gamma}_\eta k_j \mathcal{L}\left(\hat{f}^0\right) , \quad \text{for } 2 \leq \eta \leq s\,,\end{aligned} \tag{2c}$$

$$f(t_{j-1}) = \hat{f}^s\,, \tag{2d}$$

Let

$$b_\eta = \frac{P''_\eta(w_0)}{P'_\eta(w_0)^2}\,, \quad a_\eta = 1 - b_\eta P_\eta(w_0)\,.$$

For RKC, we have for $2 \leq \eta \leq s$

$$\lambda_\eta = 2\frac{b_\eta}{b_{\eta-1}} w_0\,, \quad \tilde{\lambda}_\eta = \frac{\lambda_\eta}{w_0} w_1\,,$$
$$\nu_\eta = -\frac{b_\eta}{b_{\eta-2}}\,, \quad \tilde{\gamma}_\eta = -a_{\eta-1}\tilde{\lambda}_\eta\,,$$

and $w_0 = 1 + \epsilon/s^2$, $b_0 = b_1 = 1/3$, $a_0 = 1 - b_0 w_0$, $\tilde{\lambda}_1 = b_1 w_1$, $w_1 = P'_s(w_0)/P''_s(w_0)$.

a_η and b_η are explicit for RKL (no damping)

$$b_\eta = \frac{\eta^2+\eta-2}{2\eta(\eta+1)}\,, \quad a_\eta = 1 - b_\eta\,,$$

For RKL, we have for $2 \leq \eta \leq s$

$$\lambda_\eta = \frac{2\eta-1}{\eta}\frac{b_\eta}{b_{\eta-1}} w_0\,, \quad \nu_\eta = -\frac{\eta-1}{\eta}\frac{b_\eta}{b_{\eta-2}}\,,$$

and $w_0 = 1$, $b_0 = b_1 = 1/3$, $a_0 = 1 - b_0 w_0$, $\tilde{\lambda}_1 = b_1 w_1$, $w_1 = \frac{4}{s^2+s-2}$.

For RKG a_η and b_η are explicit and read

$$b_\eta = \frac{4(\eta-1)(\eta+4)}{3\eta(\eta+1)(\eta+2)(\eta+3)}\,, \quad a_\eta = 1 - \frac{(\eta+1)(\eta+2)}{2} b_\eta\,,$$

For RKG, we have for $2 \leq \eta \leq s$

$$\lambda_\eta = \frac{2\eta+1}{\eta}\frac{b_\eta}{b_{\eta-1}} w_0\,, \quad \nu_\eta = -\frac{\eta+1}{\eta}\frac{b_\eta}{b_{\eta-2}}\,,$$

and $w_0 = 1$, $b_0 = 1$, $b_1 = 1/3$, $a_0 = 1 - b_0 w_0$, $\tilde{\lambda}_1 = w_1$, $w_1 = \frac{6}{(s+4)(s-1)}$.

RKL vs RKC

Damping $\epsilon = 2/13$ suggested in Verwer (1996). CMP = monotone with space varying coefficient.

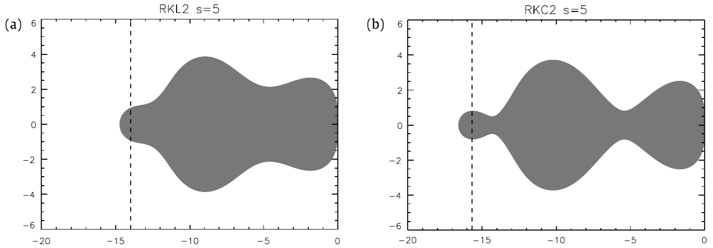


Fig. 11. (a) Shows the domain of stability in the complex plane for the $s = 5$ RKL2 scheme. (b) Shows the same for the damped $s = 5$ RKC2 scheme. The dotted line represents the maximum stable time step according to the respective method.

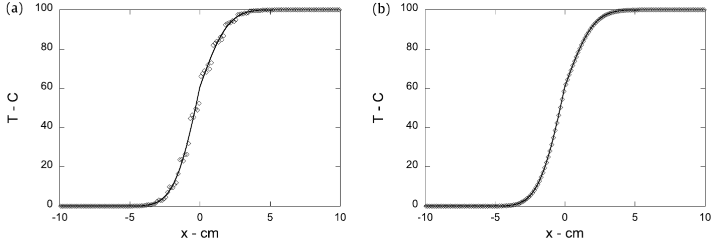
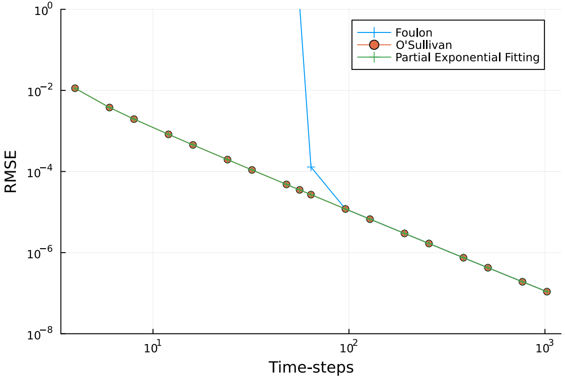


Fig. 1. (a) Shows the aluminum/copper heat conduction solution for the damped 7-stage RKC2 scheme, with damping coefficient $\varepsilon = 2/13$. (b) Shows the solution for the 7-stage RKL2 scheme.

Instabilities on the Heston PDE

$$\frac{\partial f}{\partial t} = \frac{vx^2}{2}\frac{\partial^2 f}{\partial x^2} + \rho\sigma xv\frac{\partial^2 f}{\partial x \partial v} + \frac{\sigma^2 v}{2}\frac{\partial^2 f}{\partial v^2} + (r-q)x\frac{\partial f}{\partial x} + \kappa(\theta - v)\frac{\partial f}{\partial v} - rf\,,$$

for $0 \leq t \leq T$, $x > 0$, $v > 0$, with $f(T, x, v) = F(x)$.



Convergence in time of the RKC scheme with $\epsilon = 10$, with the different choices of upwinding with $m = 100$, $n = 50$.

Different zones:

- Foulon and In't Hout (2010): three points upwinding is used at $v = 0$ and for $v > 1$.
- Le Floc'h (2019): exponential fitting is used when the Peclet number $P > 2$ and single-sided differences are used at the boundaries $x_{\min}, x_{\max}, v_{\min}, v_{\max}$.
- O'Sullivan and O'Sullivan (2013) follow Ikonen and Toivanen (2007):one-sided upwinding applied anywhere the PDE becomes convection dominated.

  > *We use central finite differences as much as possible, but when they lead to a positive codiagonal element we employ first-order accurate one-sided differences for the convection terms, that is, for the spatial first-order partial derivative terms*

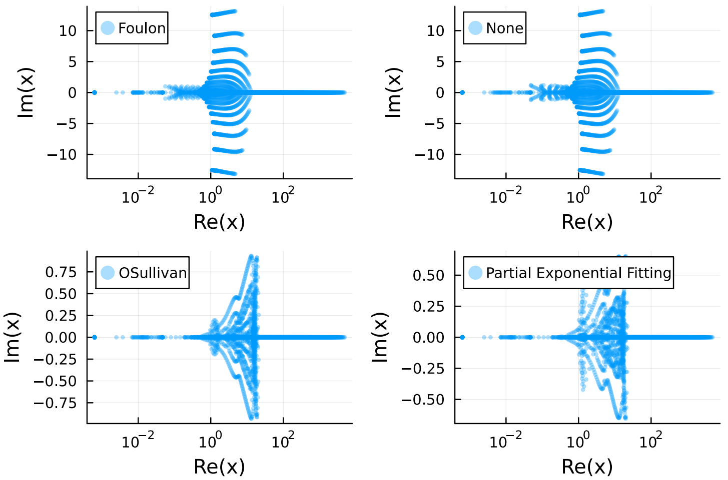
The cell Péclet number P = ratio of the advection coefficient towards the diffusion coefficient in a cell .When $P > 2$, the solution may explode.

$$P^x_{i,j}(\beta^x_{i,j}) = \frac{2h_i}{\beta^x_{i,j} v_j x_i}(r_i - q_i)\,, \quad P^v_{i,j}(\beta^v_{i,j}) = \frac{2w_j\kappa(\theta - v_j)}{\beta^v_{i,j}\sigma^2 v_j}\,.$$

$$\beta^x_{i,j} = \frac{P^x_{i,j}(1)}{2\tanh\left(\frac{P^x_{i,j}(1)}{2}\right)}\,, \quad \beta^v_{i,j} = \frac{P^v_{i,j}(1)}{2\tanh\left(\frac{P^v_{i,j}(1)}{2}\right)}\,,$$

Instabilities on the Heston PDE

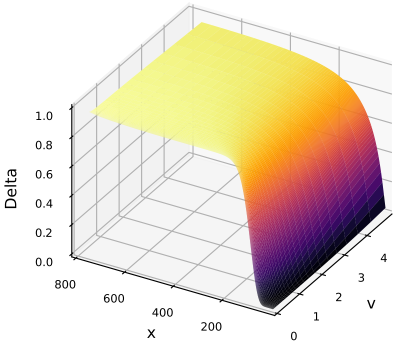
Eigenvalues

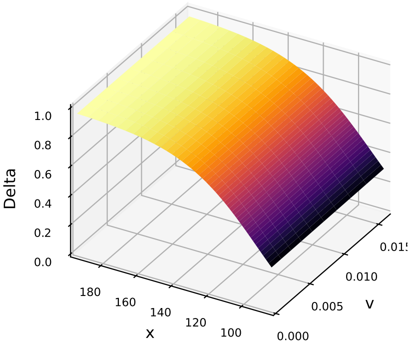


Eigenvalues of the discretization matrix with $l = 16$, $m = 100$, $n = 50$ using different upwinding choices. Note the imaginary axis range difference

Delta by forward difference for $l = 10$ time-steps and partial exponential fitting on the grid $m = 100, n = 50$

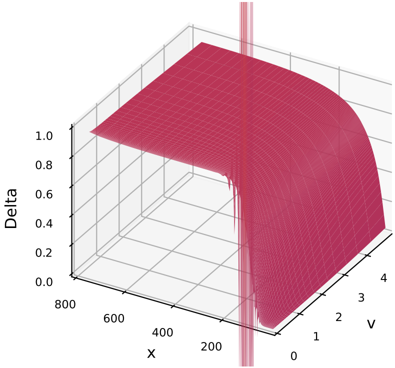
RKC with damping shift $\epsilon = 10$. No oscillations are visible near $v = 0$.

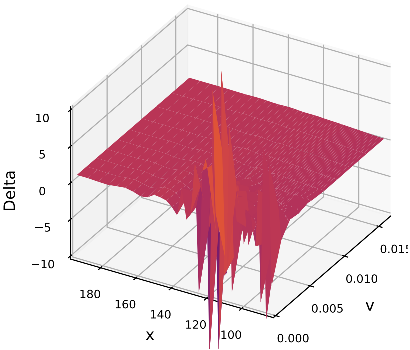




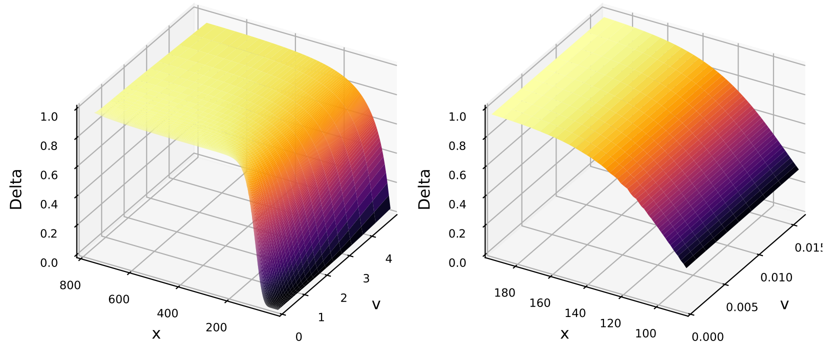
(Left: full grid, Right: zoom on small v)

RKL presents oscillations for a small number of steps. Not RKC with large damping





RKG: Very small oscillations are visible at $v = 0$.

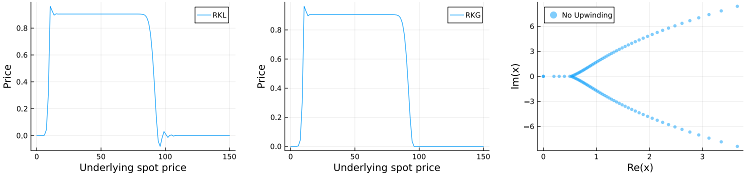


Instabilities on the Black-Scholes PDE

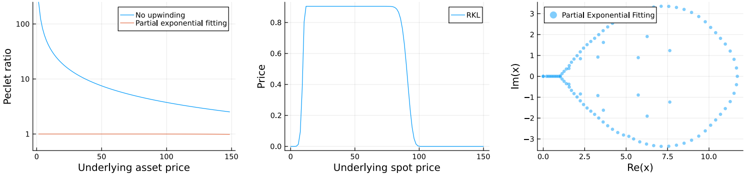
Oscillations

Small volatility $\sigma = 2\%$ and large interest rate $r = 10\%$.
Expiry barrier option which pays \$1 if $X(T)$ is between 10 and 100 and zero otherwise, $T = 1$. $l = 100$ time-steps on a uniform grid with $m = 100$ space steps.

No upwinding (RKL, RKG or TR-BDF2, Eigenvalues)



Exponential fitting (P, RKL, Eigenvalues)

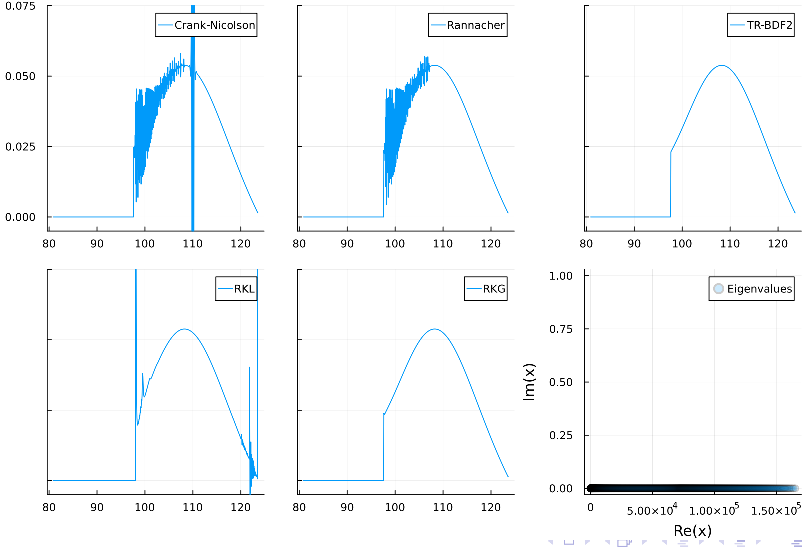


upwinding important, solves the two oscillations.

Instabilities on the Black-Scholes PDE

Gamma of American Put - The need for damping

100 time-steps and 1000 space steps with $x_{\min} = 80.89$ and $S_{\max} = 123.59$ (3 standard deviations with $\sigma = 10\%$, $r = 1\%$). American option with strike $K = 110$, $T = 0.5$. Peclet $P < 10^{-3}$



Ghost points

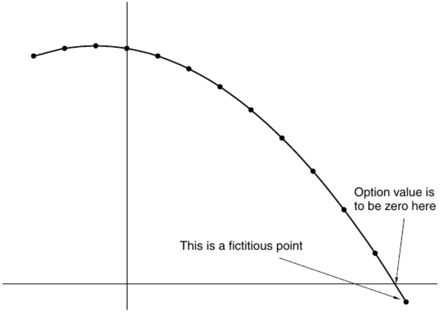


Figure 77.6 A fictitious point, introduced to ensure accuracy in a barrier option boundary condition.

This condition can be approximated by ensuring that the straight line connecting the option values at the two grid points straddling the barrier has the value f at the barrier. Then a good discrete version of this boundary condition is

$$V_I^k = \frac{1}{\alpha}\left(f - (1-\alpha)V_{I-1}^k\right)$$

where

$$\alpha = \frac{S_u - (I-1)\delta S}{\delta S}.$$

This is accurate to $O(\delta S^2)$, the same order of accuracy as in the approximation of the S

Black-Scholes PDE, uniform discretization. When $r = q = 0$,

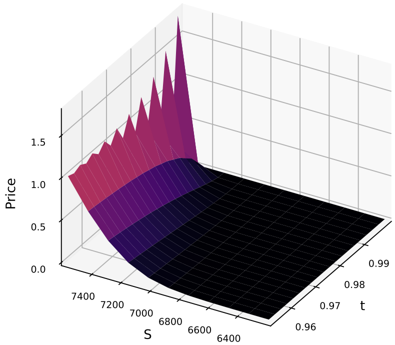
$$\delta t \leq \frac{4\delta S^2}{\sigma_k^2 S_{u-1}^2 \left(3 + \frac{S_u - L^+}{(L^+ - S_{u-1})}\right)}\,.$$

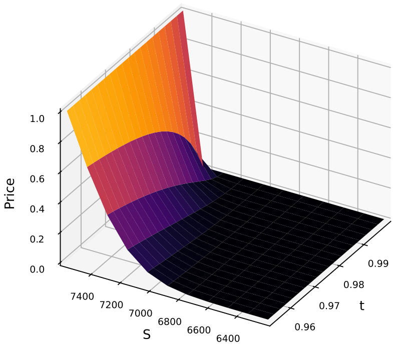
Increasingly stringeant as the grid point S_{u-1} moves towards L^+.
Let $\epsilon = L^+ - S_{u-1}$, we have

$$\frac{4\delta S^2}{\sigma_k^2 S_{u-1}^2 \left(3 + \frac{\delta S - \epsilon}{\epsilon}\right)} = \frac{4\delta S}{\sigma_k^2 S_{u-1}^2}\epsilon + \mathcal{O}(\epsilon^2)\,.$$

What about implicit schemes?

Price of a one-touch option obtained by the Crank-Nicolson scheme on the finite difference grid, for $M = 100$ space-steps and $N = 400$ time-steps, close to the expiry and the barrier level. Uniform with ghost point vs. Stretched, barrier on grid





TR-BDF2 no oscillations at all (excluding first two time-steps).

- STS are interesting in many practical cases
- upwinding important but lack of damping may be a concern.
- Ghost point technique should not be used with STS.
- RKL, RKG interesting properties, but would those also be achievable with a proper choice of damping coefficient in RKC?
- Don't use too many stages?

References

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