Valuation of Equity and Debt with finite maturity using local time

Yerkin Kitapbayev (Khalifa University, Abu Dhabi)

joint work with Jerome Detemple (Boston U)

International Conference on Computational Finance
CWI, Amsterdam, 2-5 April 2024

Objectives of the paper

- Objectives:
  - We examine optimal debt contracts in a dynamic model with a finite maturity date and a continuous coupon payments.
  - The valuation formulas for the equity and debt values.
  - The value of equity has an early default premium representation where the endogenous default boundary solves a recursive integral equation.
  - The debt value is given by the representation that involve a local time term.
  - We develop a numerical algorithm that employs these characterizations.
- Context:
  - Classical paper by Leland (1994) and subsequent literature assume perpetual contracts for tractability reasons.
  - Finite maturity contracts involve time-dependent problems.

Model

- Let us assume that the firm at time t produces the cash flow at rate X_t that follows
$$dX_t/X_t = \mu dt + \sigma dW_t \tag{1}$$
under the risk-neutral measure Q, where $\mu = r - \delta < r$ is the risk-neutral drift, δ is the payout ratio, and $\sigma > 0$ is the volatility.
- Now we consider the coupon-bearing bond that pays continuously coupons at rate c until the maturity date $T > 0$ at which there is principal repayment of $P > 0$ and after that it becomes all-equity firm with the after-tax value
$$U(X_T) = (1-\theta)\mathbb{E}_T\left[\int_T^\infty e^{-r(s-T)}X_s ds\right] = (1-\theta)X_T/\delta. \tag{2}$$
- The parameter $\theta \in (0,1)$ is the tax rate, and we suppose the debt is not callable.

Equity value

- Then the equity holders solve the following optimal default problem

$$E(X_0; c, P) = \sup_{\tau \in [0,T]} \mathbb{E}\Big[\int_0^\tau e^{-rt}(1-\theta)(X_t - c)dt \\ + e^{-rT}\left(U(X(T)) - P\right)^+ I_{\tau=T} \Big] \quad (3)$$

  where supremum is taken over the set of stopping times with values in $[0, T]$ and $I_{\tau=T}$ is the indicator of the event the default has not happened before T.
- The intuition behind (3) is the following: until default time τ the equity holders collect cash flows at X_t and pay coupon c. If they do not default before T, they have option to extend the ownership of firm and get the present value of future cash flows $U(X_T)$ but for this the face value P must be paid. If $\tau < T$, the firm's assets are transferred from equity holders to debt holders.

Debt value

- Given optimal default (random) time τ_d, we have that the debt value is given by

$$D(X_0; c, P) = \mathbb{E}\Bigg[\int_0^{\tau_d} e^{-rt} c dt + e^{-r\tau_d}(1-\alpha)U(X_\tau) I_{\tau_d < T} \qquad (4)$$
$$+ e^{-rT} \left(P I_{\Omega/A} + (1-\alpha)U(X_T) I_A \right) \cdot I_{\tau_d = T} \Bigg]$$

where α is the bankruptcy cost and $A := \{U(X_T) < P\}$ is the event of default at time T.

Equity value

- We postulate that there exists an optimal default boundary b on $[0, T]$. Then given the boundary b, the equity value can be computed as follows.
- The equity value $E(t,x)$ and free boundary $b(t)$ solve

$$E_t + \mu x E_x + \frac{1}{2}\sigma^2 x^2 E_{xx} + (1-\theta)(x-c) = rE, \quad x > b(t), t \in [0, T)$$

$$E(t, b(t)) = 0, \quad t \in [0, T) \tag{5}$$

$$E_x(t, b(t)) = 0, \quad t \in [0, T) \tag{6}$$

$$E(T, x) = (U(x) - P)^+, \quad x > 0 \tag{7}$$

$$b(T) = \min(c, P/\kappa) \tag{8}$$

$$E(t, x) = 0, \quad t \in [0, T), x \le b(t). \tag{9}$$

Equity value

- The benchmark strategy is to wait and not default until T so that the associated value at time $t \in [0, T)$ is given by

$$\begin{aligned} & E_w(t, X_t; c, K) \\ = \; & \mathbb{E}_t \left[\int_t^T e^{-r(u-t)}(1-\theta)(X_u - c)du + e^{-r(T-t)}\kappa(X_T - K)^+ \right] \\ = \; & (1-\theta)\mathbb{E}_t \left[\int_t^T (e^{(\mu-r)(u-t)}X_t - ce^{-r(u-t)})dt \right] + \kappa C(t, X_t) \\ = \; & (1-\theta)\left(X_t(1 - e^{(\mu-r)(T-t)}) - c(1 - e^{-r(T-t)}) \right) + \kappa C(t, X_t) \end{aligned}$$

where $\kappa = (1-\theta)/(r-\mu)$, $K = P/\kappa$ and $C(t, X_t)$ is the time-t European call option price under Black-Scholes model with maturity T and strike K.

Equity value

- This strategy is sub-optimal and the early default premium can be found as

$$\pi(t, X_t; b, c) = \mathbb{E}_t \left[\int_t^T e^{-r(u-t)} (1-\theta)(c - X_u) I(X_u \leq b(u)) du \right] \quad (10)$$

- The intuition is the following: when it is optimal to default, i.e., $I(X_u \leq b(u))$ the benefits of default are $(1-\theta)(c - X_u)$. We then discount these benefits and integrate over $[t, T]$, and take expected value.

EDP formula

- The equity value is given by early default premium formula

$$
\begin{aligned}
& E(t, X_t; c, K) \\
= \; & E_w(t, X_t; c, K) + \pi(t, X_t; b, c) \\
= \; & \kappa C(t, X_t) + \mathbb{E}_t \left[\int_t^T e^{-r(u-t)} (1-\theta)(X_u - c) I(X_u > b(u)) du \right] \\
= \; & \kappa C(t, X_t) + (1-\theta) X_t \int_t^T e^{(\mu - r)(u-t)} N(d^+(X_t, b(u), u-t)) du \\
& -(1-\theta) c \int_t^T e^{-r(u-t)} N(d^-(X_t, b(u), u-t)) du
\end{aligned}
$$

where N is the standard normal cdf and

$$
d^{\pm}(x, y, v) = \frac{1}{\sigma \sqrt{v}} \left(\log \frac{x}{y} + \left(\mu \pm \frac{\sigma^2}{2} \right) v \right)
$$

Optimal default boundary

- To characterize the default boundary, we can recall the value matching condition at $X_t = b(t)$, i.e.,

$$E(t, b(t); c, K) = 0 \tag{11}$$

and hence we get the Volterra integral equation for $b(t)$

$$0 = \frac{1}{r-\mu} C(t, b(t)) + b(t) \int_t^T e^{(\mu-r)(u-t)} \phi^+(b(t), b(u), u-t) du$$
$$- c \int_t^T e^{-r(u-t)} \phi^-(b(t), b(u), u-t) du$$

for $t \in [0, T)$ with $b(T-) = \min(c, K)$. Once we solve the integral equation, we obtain the equity value using the EDP above.

Optimal default boundary

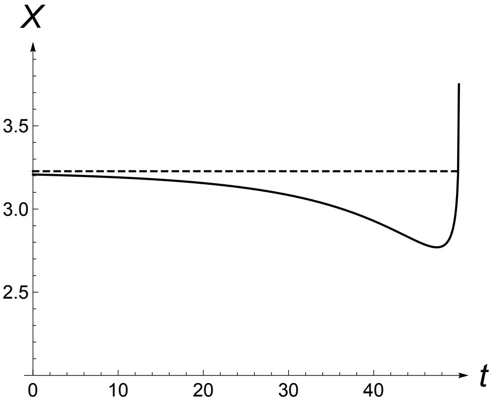

Figure: This figure displays the default boundary. Parameters are $\theta = 0.2$; $r = 0.05$; $\mu = 0.02$; $P = 100$; $c = 12$; $\sigma = 0.3$; $T = 50$.

Equity value

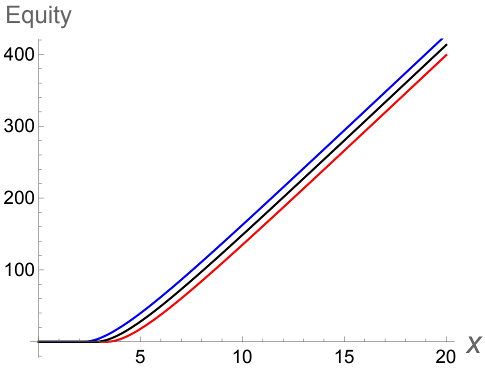

Figure: This figure illustrates the effect of c ($c = 8$ (blue), $c = 12$ (black), $c = 16$ (red)) on the equity value. Parameters are $r = 0.05$, $\mu = 0.02$, $\sigma = 0.3$, $P = 100$, $c = 12$, $T = 5$ years, $\theta = 0.2$.

PDE for debt value

- Given the fixed boundary $b(t)$, the debt value $D(t,x)$ satisfies

$$D_t + \mu x D_x + \frac{1}{2}\sigma^2 x^2 D_{xx} + c = rD, \quad x > b(t), t \in [0, T) \tag{12}$$
$$D(t, b(t)) = (1-\alpha)U(b(t)), \quad t \in [0, T) \tag{13}$$
$$D(t,x) = (1-\alpha)U(x), \quad t \in [0, T), \;\; x \le b(t) \tag{14}$$
$$D(T,x) = P \cdot I_{U(x)\ge P} + (1-\alpha)U(x) I_{U(x)<P}, \quad x > 0 \tag{15}$$
$$b(T) = \min(c, P/\kappa). \tag{16}$$

where U is an all-equity firm value

$$U(x) = (1-\theta)x/\delta, \quad x > 0. \tag{17}$$

Debt value

- Now let us apply the local time-space formula for $e^{-r(T-t)}D(T,X_T)$

$$
\begin{aligned}
&e^{-r(T-t)}D(T,X_T) \qquad (18)\\
&\quad = D(t,x) - \int_t^T e^{-r(u-t)} c I(X_u > b(u)) du \\
&\qquad - \int_t^T e^{-r(u-t)} (1-\alpha)(1-\theta) X_u I(X_u \le b(u)) du \\
&\qquad + M_t + \frac{1}{2}\int_t^T e^{-r(u-t)} (D_x(u,b(u)+) - (1-\alpha)\kappa) d\ell_u^b(X)
\end{aligned}
$$

where M is the martingale and $\ell^b(X)$ is the local time of X at the curve b, given by

$$
\ell_t^b(X) = Q - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t I\left(b(s) - \varepsilon < X_s < b(s) + \varepsilon\right) d\langle X, X\rangle_s.
$$

Debt value

- Now by taking the expected value on both sides, using the optional sampling theorem, and inserting $x = b(t)$, we obtain

$$
\begin{aligned}
&\mathbb{E}_{t,b(t)}[e^{-r(T-t)}D(T, X_T)] \qquad (19)\\
&=(1-\alpha)(1-\theta)b(t)/\delta - \mathbb{E}_{t,b(t)}\left[\int_t^T e^{-r(u-t)} cI(X_u > b(u))du\right]\\
&\quad - \mathbb{E}_{t,b(t)}\left[\int_t^T e^{-r(u-t)}(1-\alpha)(1-\theta)X_u I(X_u \le b(u))du\right]\\
&\quad + \frac{1}{2}\int_t^T e^{-r(u-t)}(D_x(u, b(u)+) - \frac{(1-\alpha)(1-\theta)}{\delta})d\mathbb{E}_{t,b(t)}[\ell_u^b(X)]
\end{aligned}
$$

for $t \in [0, T]$.

- This gives us linear Volterra equation of the first kind for $D_x(t, b(t)+)$ which we can solve numerically by backward induction.

Debt value

- We note that $d\mathbb{E}_{t,x}[\ell_u^b(X)] = K(t,x;u)du$, where

$$K(t,x;u) = \varphi\left(-\frac{\log\left(b(u)/x\right) - \left(r-\delta-\frac{\sigma^2}{2}\right)(u-t)}{\sigma\sqrt{u-t}}\right)\frac{\sigma b(u)}{\sqrt{u-t}} \tag{20}$$

and $\varphi(\cdot)$ is the probability density of function of $N(0,1)$,

Delta of debt value at $b(t)+$

- Alternatively, we write

$$
\begin{aligned}
D(t,x) = & F(t,x) \\
& - \frac{1}{2}\int_t^T e^{-r(u-t)}\big(D_x(u,b(u)+) - (1-\alpha)(1-\theta)/\delta\big) \\
& \times K(t,x;u)du
\end{aligned} \tag{21}
$$

for known function F.

- Taking the derivative with respect to x and evaluating this expression at $x = b(t)+$ gives linear Volterra equation of the second kind

$$
\begin{aligned}
D_x(t,b(t)+) = & F_x(t,b(t)+) \\
& - \frac{1}{2}\int_t^T e^{-r(u-t)}\big(D_x(u,b(u)+) - (1-\alpha)(1-\theta)/\delta\big) \\
& \times K_x(t,b(t)+;u)du.
\end{aligned} \tag{22}
$$

Delta of debt value at $b(t)+$

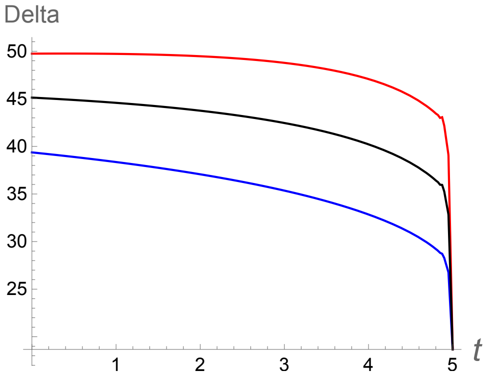

Figure: This figure plots the debt delta $D_x(t, b(t)+)$ along the boundary b as the function of t for $c = 8$ (blue), $c = 12$ (black), $c = 16$ (red). Parameters are $r = 0.05$, $\mu = 0.02$, $\sigma = 0.3$, $P = 100$, $T = 5$ years, $\theta = 0.2$, $\alpha = 0.3$

Debt value

- Once we recover $D_x(t, b(t)+)$, we can compute the debt value as

$$
\begin{aligned}
&D(t,x) \\
=&\mathbb{E}_{t,x}[e^{-r(T-t)}D(T,X_T)] + \mathbb{E}_{t,x}\left[\int_t^T e^{-r(u-t)} c I(X_u > b(u)) du\right] \\
&+ \mathbb{E}_{t,x}\left[\int_t^T e^{-r(u-t)}(1-\alpha)(1-\theta) X_u I(X_u \le b(u)) du\right] \\
&- \frac{1}{2}\int_t^T e^{-r(u-t)}(D_x(u, b(u)+) - (1-\alpha)(1-\theta)/\delta) K(t,x;u) du
\end{aligned}
$$

Debt value

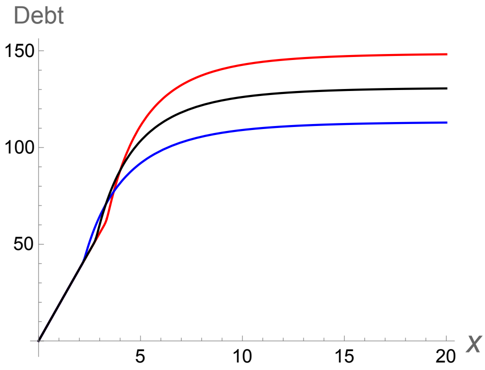

Figure: This figure illustrates the effect of c ($c = 8$ (blue), $c = 12$ (black), $c = 16$ (red)) on the debt value. Parameters are $r = 0.05$, $\mu = 0.02$, $\sigma = 0.3$, $P = 100$, $c = 12$, $T = 5$ years, $\theta = 0.2$, $\alpha = 0.3$.

Barrier options

- Similar idea has been applied by Mijatovic (2010) for pricing of barrier options.
- Let us consider the down-and-out barrier call option under Black-Scholes model with the value function V, strike K and the barrier $H(t) < K$

$$V_t + \mu x V_x + \frac{1}{2}\sigma^2 x^2 V_{xx} = rV, \quad x > H(t), t \in [0, T) \tag{23}$$

$$V(t, H(t)) = 0, \quad t \in [0, T) \tag{24}$$

$$V(t, x) = 0, \quad t \in [0, T), x \leq H(t) \tag{25}$$

$$V(T, x) = (x - K)^+, \quad x > 0. \tag{26}$$

Barrier options

- Now let us apply the local time-space formula for $e^{-r(T-t)}V(T,X_T)$

$$
\begin{aligned}
&e^{-r(T-t)}V(T,X_T) \\
&\quad = V(t,x) + \int_t^T e^{-r(u-t)}(\mathbb{L}-r)V(u,X_u)du \\
&\quad\quad + M_T + \frac{1}{2}\int_t^T e^{-r(u-t)}V_x(u,H(u)+)d\ell_u^H(X)
\end{aligned} \tag{27}
$$

where M is the martingale and $\ell^H(X)$ is the local time at H.

- Now by taking the expected value on both sides, using the optional sampling theorem, and inserting $x = H(t)$, we obtain

$$
\begin{aligned}
&\mathbb{E}_{t,H(t)}[e^{-r(T-t)}(X_T-K)^+] \\
&\quad = \frac{1}{2}\int_t^T e^{-r(u-t)}V_x(u,H(u)+)d\mathbb{E}_{t,H(t)}[\ell_u^H(X)]
\end{aligned} \tag{28}
$$

for $t \in [0,T]$.

Barrier options

- This gives us linear Volterra equation of the first kind for $V_x(t, H(t)+)$ which we can solve numerically by backward induction.

- Once we recover $V_x(t, H(t)+)$, we can compute the barrier option premium as

$$
\begin{aligned}
V(t,x) =& \mathbb{E}_{t,x}[e^{-r(T-t)}(X_T - K)^+] \\
&- \frac{1}{2}\int_t^T e^{-r(u-t)} V_x(u, H(u)K(t,x;u)du
\end{aligned}
$$

References

References:

- Black, F., and J.C. Cox. 1976. Valuing Corporate Securities: Some Effects of Bond Indenture Provisions. *Journal of Finance* 31(2): 351—367.
- Carr, P., R. Jarrow, and R. Myneni. 1992. Alternative Characterizations of American Put Options. *Mathematical Finance* 2(2): 87–106.
- Itkin, A., Lipton, A., and D. Muravey. Generalized Integral Transforms in Mathematical Finance. WSPC, Singapore, 2021.
- Leland, H. 1994. Corporate Debt Value, Bond Covenants, and Optimal Capital Structure. *Journal of Finance* 49(4): 1213–1252.
- Merton R.C. 1974. On the Pricing of Corporate Debt: the Risk Structure of Interest Rates. *Journal of Finance* 29(2): 449–470.
- Mijatovic, A. 2010. Local time and the pricing of time-dependent barrier options. *Finance & Stochastics* 14 (1): 13-–48

Thank you!