VIX Option pricing for non-parameter Heston stochastic local volatility model[1]

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[1]joint work with Jiaxin Gong and Junmei Ma.

Outline

Option and its models

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- Call/Put option: holder has the *right* to buy/sell an underlying asset (S_t) at a predetermined price (Strick price K) in future (Maturity T) from the writer.
- European/American Option
- Black-Sholes Model (1979)

$$dS_t = rS_t dt + \sigma S_t dW,$$

where r is the risk-free interest rate, σ is the volatility of S_t and dW is a Brownian motion

- Heston stochastic volatility (SV) model (1991)
- Stochastic local volatility (SLV) model (1999)

Non-parametric Heston-Duprie SLV model

. The underlying asset S_t satisfying following stochastic differential equation(SDE):

$$\begin{cases} dS_t = rS_t dt + L(t, S_t)\sqrt{v_t} S_t dW_t^1, \\ dv_t = \kappa(\theta - v_t)\, dt + \sigma_v \sqrt{v_t} dW_t^2. \end{cases} \tag{2.1}$$

where v_t is the variance of S_t, r is the risk-free interest rate; κ is the mean-reversion speed of the variance; θ is the long-term mean variance and σ_v is the volatility of variance. The correlation between two driven Brownian motions is ρ.

Remark: *It collapses to many popular SV or SLV models when the leverage function* $L(t, S_t)$ *defined in some special forms.*

Leverage function $L(t, S_t)$

. Without assuming any particular from, the leverage function satisfies

$$L^2(t,K) = \frac{\frac{\partial C(t,K)}{\partial t} + rK\frac{\partial C(t,K)}{\partial K}}{\frac{1}{2}K^2\frac{\partial^2 C(t,K)}{\partial K^2}\mathbb{E}\left[v_t \mid S_t = K\right]} := \frac{\sigma_{LV}^2(t,K)}{\mathbb{E}\left[v_t \mid S_t = K\right]},$$

where $C(t, K)$ is the European call option matured at t with strike price K, and $\sigma_{LV}(t, K)$ is called Duprie's local volatility, which is in the form of

$$\sigma_{LV}^2(t,K) = \frac{\sigma_I^2(t,K) + 2t\sigma_I(t,K)\left(\frac{\partial \sigma_I}{\partial t} + rK\frac{\partial \sigma_I}{\partial K}\right)}{\left(1 + d_1 K\sqrt{t}\frac{\partial \sigma_I}{\partial K}\right)^2 + K^2\sigma_I(t,K)t\left(\frac{\partial^2 \sigma_I}{\partial K^2} - d_1\sqrt{t}\left(\frac{\partial \sigma_I}{\partial K}\right)^2\right)},$$

where $\sigma_I(t, K)$ is the implied volatility and $d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma_I^2(t,K)\right)t}{\sigma_I(t,K)\sqrt{t}}$.

Stochastic volatility inspired (SVI) model

. Given a maturity T_n, the SVI model of implied volatility $\sigma_I(t,K)$ is

$$\sigma_I^{SVI}(t,x) = \sqrt{\frac{\alpha_n + \beta_n \left[\rho_n (x - m_n) + \sqrt{(x - m_n)^2 + \zeta_n}\right]}{t}},$$

where $\alpha_n, m_n \in \mathcal{R}, \beta_n \geq 0, |\rho_n| < 1, \zeta_n > 0$, and $x = \ln\left(\frac{K}{S_0}\right)$.

SVI curves of SPX on Feb 21, 2018.

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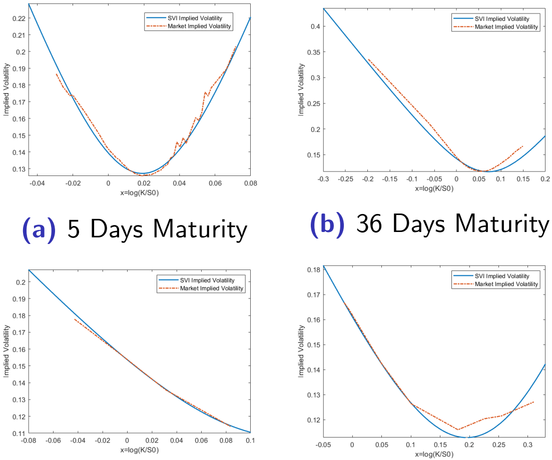

(a) 5 Days Maturity

(b) 36 Days Maturity

(c) 160 Days Maturity

(d) 331 Days Maturity

Willow Tree Method

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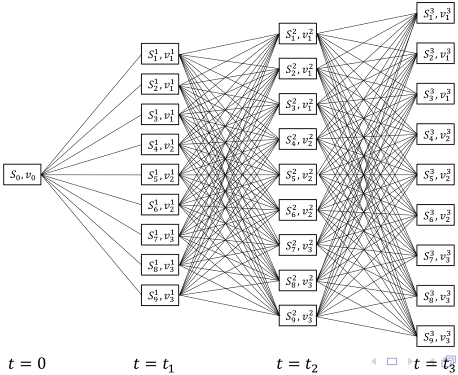
- Proposed by [Curran, 2001], but improved by [Xu et al., 2013].
- Applicable to various continuous models, such as diffusion models and Levy models, and discrete models, such as GARCH models, in option pricing and risk management.



Willow Tree Construction

. Two main steps: tree node pairs (S_i^n, v_{i1}^n), and transition probability $\mathbf{P}^n = [p_{ij}^n]$, $i, j = 1, 2, \cdots, m_v \cdot m_x$, $n = 1, 2 \cdots, N$.

- Let $X_t = \ln S_t$, we have

$$\begin{cases} dX_t = \left(r - \frac{1}{2} L^2\left(t, e^{X_t}\right) v_t\right) dt + L\left(t, e^{X_t}\right) \sqrt{v_t} dW_t^1 \\ dv_t = \kappa\left(\theta - v_t\right) dt + \sigma_v \sqrt{v_t} dW_t^2. \end{cases} \tag{3.2}$$

- Given v_t, the first four moments of X_t can be evaluated

$$\mathbb{E}\left[(X_t)^d \mid v_t\right] = \sum_{k=0}^{\mathcal{K}} \left\{\mathcal{L}^k\left[(X_t)^d\right]\right\}_{t=0} \frac{t^k}{k!} + R_{\mathcal{K}+1}, \quad d = 1, 2, 3, 4,$$

where the operator $\mathcal{L}$ and residual are in the form of

$$\mathcal{L} = \left(r - \frac{1}{2} L^2\left(t, e^{X_t}\right) v_t\right) \frac{\partial}{\partial X_t} + \frac{1}{2} L^2\left(t, e^{X_t}\right) v_t \frac{\partial^2}{\partial X_t^2},$$

and

$$R_{K+1} = \frac{t^{\mathcal{K}+1}}{(\mathcal{K}+1)!} \mathbb{E}\left\{\mathcal{L}^k\left[(X_\xi)^d\right]\right\}, \quad 0 \leq \xi \leq t.$$

Tree nodes construction $(X_i^n, v_{i_1}^n)$

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- Given v_t following a CIR model, the first four moments of v_t can be evaluated analytically[Wang and Xu, 2018].
- At t_n, m_v discrete values of v_t can be generated by the Johnson curve [Johnson, 1949] to match the first four moments of v_{t_n}.
- Given $v_{i_1}^n$, the first four moments of X_t at t_n can be evaluated.
- Given $v_{i_1}^n$, m_x discrete values of X_t can be generated by the Johnson curve to match the first four conditional moments of X_{t_n}

Transition probability matrix $\mathbf{P}^n = [p_{ij}^n]$

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- The transition probability p_{ij}^n between $(X_i^n, v_{i_1}^n)$ and $(X_j^{n+1}, v_{j_1}^{n+1})$ is a joint conditional probability

$$p_{ij}^n = \mathbb{P}\left(A_j^{n+1} \leq X_{t_{n+1}} \leq A_{j+1}^{n+1}, a_{j_1}^{n+1} \leq v_{t_{n+1}} \leq a_{j_1+1}^{n+1} \mid (X_i^n, v_{i_1}^n)\right),$$

  where $A_j^{n+1} = (X_{j-1}^{n+1} + X_j^{n+1})/2$, $a_{j_1}^{n+1} = (v_{j_1-1}^{n+1} + v_{j_1}^{n+1})/2$, $a_1^{n+1} = A_1^{n+1} = -\infty$, and $a_{m_v+1}^{n+1} = A_{m_v \cdot m_x+1}^{n+1} = +\infty$.

- Define a bivariate normally distributed random variable $(Z, \tilde{Z})$, where $Z \sim N(0,1)$, $\tilde{Z} \sim N(0,1)$ and the correlation ρ.
- Define $\Delta X^{n+1} = X_{t_{n+1}} - X_i^n$ and $\Delta v^{n+1} = v_{t_{n+1}} - v_{i_1}^n$, we have

$$\Delta X^{n+1} = \left(r - \frac{1}{2}L^2(t_n, X_i^n)\, v_{i_1}^n\right)\Delta t + L(t_n, X_i^n)\sqrt{v_{i_1}^n}\sqrt{\Delta t}Z$$

  and

$$\Delta v^{n+1} = \kappa\left(\theta - v_{i_1}^n\right)\Delta t + \sigma_v\sqrt{v_{i_1}^n}\sqrt{\Delta t}\tilde{Z}.$$

The transition probability p_{ij}^n can be estimated as

$$p_{ij}^n = \mathbb{P}\left(C_j^{n+1} \leq Z \leq C_{j+1}^{n+1}, c_{j_1}^{n+1} \leq \tilde{Z} \leq c_{j_1+1}^{n+1}\right), \qquad (3.3)$$

where $C_j^{n+1} \equiv \frac{\left(A_j^{n+1} - X_i^n - \left(r - \frac{1}{2}L^2\left(t_n, X_i^n\right)v_{i_1}^n\right)\Delta t\right)}{L\left(t_n, X_i^n\right)\sqrt{v_{i_1}^n}\sqrt{\Delta t}}$, and
$c_{j_1}^{n+1} \equiv \frac{\left(a_{j_1}^{n+1} - v_{i_1}^n - \kappa(\theta - v_{i_1}^n)\Delta t\right)}{\sigma_v\sqrt{v_{i_1}^n}\sqrt{\Delta t}}$.
Remark: *The transition probability is evaluated by the joint cumulative distribution of* $(Z, \tilde{Z})$.

Evaluate Leverage function $L(t, K)$

Given the willow tree until t_n, we evaluate $\mathbb{E}[v_{t_n}|S_{t_n} = S_i^n]$ for p_{ij}^n from $(S_i^n, v_{i_1}^n)$ to $(S_j^{n+1}, v_{j_1}^{n+1})$. Define $\mathbf{q}^n = (\mathbf{q}^0)^T \cdot \mathbf{P}^1 \cdot \mathbf{P}^2 \cdots \mathbf{P}^{n-1}$ and divide the range S_{t_n} into I mutually exclusive bins $(b_1^n, b_2^n], (b_2^n, b_3^n], \cdots, (b_I^n, b_{I+1}^n]$ with $b_1^n \geq 0$ and $b_{I+1}^n \leq \infty$, we have

$$\mathbb{E}\left[v_{t_n} \mid S_{t_n} = S_i^n\right] \approx \mathbb{E}\left[v_{t_n} \mid S_{t_n} \in (b_i^n, b_{i+1}^n]\right] \approx \frac{\mathbb{E}\left[v_{t_n}\mathbf{1}_{\{S_{t_n}\in(b_i^n, b_{i+1}^n]\}}\right]}{\mathbb{P}\left[S_{t_n} \in (b_i^n, b_{i+1}^n]\right]}, \tag{3.4}$$

Given the willow tree till t_n, (3.4) can be evaluated as

$$\mathbb{E}\left[v_{t_n} \mid S_{t_n} = S_i^n\right] \approx \frac{\mathbb{E}\left[v_{t_n}\mathbf{1}_{S_{t_n}\in(b_i^n, b_{i+1}^n]}\right]}{\mathbb{P}\left[S_{t_n} \in (b_i^n, b_{i+1}^n]\right]} \approx \frac{\sum_{j=1}^{m_x\times m_v} \tilde{q}_j^n \tilde{v}_j^n \mathbf{1}_{\tilde{S}_j^n\in\left(b_i^n, b_{i+1}^n\right]}}{\sum_{j=1}^{m_x\times m_v} \tilde{q}_j^n \mathbf{1}_{\tilde{S}_j\in\left(b_i^n, b_{i+1}^n\right]}},$$

where $[\tilde{q}_j^n]$ is the sorted vector of $\mathbf{q}^n$ according to $\tilde{S}_j^n$.

Option pricing on willow tree

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- European call option
  - $V_i^N = \max\{S_i^N - K, 0\}$, for $i = 1, 2, \cdots .m$, $n = 1, 2, \cdots, N-1$
  - $V_i^n = e^{-r\Delta t} \sum_{j=1}^{m} p_{ij}^n V_j^{n+1}$
  - $V(S_0, 0) = e^{-r\Delta t} \sum_{i=1}^{m} q_i^1 V_i^1$
- American put option
  - $V_i^N = \max\{K - S_i^N, 0\}$, for $i = 1, 2, \cdots .m$, $n = 1, 2, \cdots, N-1$
  - $V_i^n = \max\left\{K - S_i^n, e^{-r\Delta t} \sum_{j=1}^{m} p_{ij}^n V_j^{n+1}\right\}$
  - $V(S_0, 0) = \max\left\{K - S_0, e^{-r\Delta t} \sum_{i=1}^{m} q_i^1 V_i^1\right\}$

VIX option pricing

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- VIX (CBOE 30-day volatility index, "fear index") option is very popular, average daily trading volume 600,000+ in Jan. 2021.
- Definition of VIX

$$\mathrm{VIX}_T^2(\tau) = \frac{2}{\tau}\mathbb{E}_T^Q\left[\int_T^{T+\tau} \frac{\mathrm{d}S_u}{S_u} - \mathrm{d}\left(\log S_u\right)\right] \times 100^2. \quad (3.5)$$

- Given S_t following the Heston-Dupire model, the VIX on willow tree can be defined as

$$\mathrm{VIX}_i^N = \sqrt{\frac{\Delta\tau \times 100^2}{\tau} \sum_{n=N}^{N'-1} \mathbb{E}^Q\left[L^2\left(t_n, S_{t_n}\right) v_{t_n} \mid \left(S_i^N, v_{i_1}^N\right)\right]}, \quad (3.6)$$

Convergence Analysis

. Define the European call option $U(t, S_t, v_t)$, it satisfies following partial differentiable equation (PDE)

$$\begin{aligned} &\frac{\partial U}{\partial t} + rS\frac{\partial U}{\partial S} + \frac{1}{2}vS^2L^2(t,S)\frac{\partial^2 U}{\partial S^2} + \frac{1}{2}v\sigma_v^2\frac{\partial^2 U}{\partial v^2} \\ &+[\kappa(\theta - v)]\frac{\partial U}{\partial v} + \rho\sigma_v vSL(t,S)\frac{\partial^2 U}{\partial S \partial v} = rU. \end{aligned} \tag{4.7}$$

Theorem

Given (S_t, v_t) following the Heston-Dupire model, the computed European option price by the backward induction on the 2-D willow tree converges to the solution of (4.7) as $\Delta t \to 0$ where $\Delta t = T/N$.

Parameter setting

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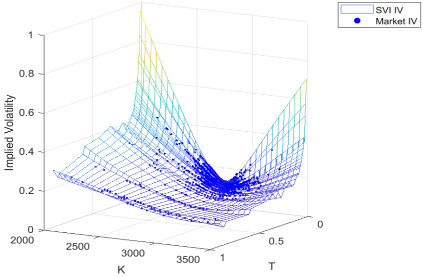
- Heston-Duprie model parameters : risk-free interest rate $r = 5\%$; the mean reversion speed $\kappa = 2.8$; the mean reversion level $\theta = 0.12$; the volatility of volatility $\sigma_v = 0.05$.
- SVI surface from implied volatilities of S&P 500 index options on Feb 21, 2018.



Pricing results

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	European call option						
K	2600	2650	2700	2750	2800	2850	CPU time
upper 95% C.I.	231.8469	193.8914	158.4179	125.9669	97.1430	72.4637	
MC	230.6486	192.7691	157.3784	125.0163	96.2859	71.7023	204.20 seconds
lower 95% C.I.	229.4503	191.6468	156.3389	124.0657	95.4289	70.9408	
WT	231.6441	193.3343	157.6440	125.0565	96.0898	71.2122	9.99 seconds
	American put options						
K	2600	2650	2700	2750	2800	2850	CPU time
upper 95% C.I.	229.8407	192.0267	156.6845	124.4984	96.1622	71.6535	
MC	228.2795	190.5754	155.3539	123.2926	95.0768	70.7064	2.31 hours
lower 95% C.I.	226.7183	189.1241	154.0233	122.0867	93.9913	69.7592	
WT	229.3915	191.1906	155.5285	123.0019	94.1704	69.5864	10.1876 seconds
	VIX call options						
K	8	10	12	14	16	18	CPU time
upper 95% C.I.	8.8288	6.9071	5.6283	4.7453	4.0812	3.5615	
MC	8.4681	6.5469	5.2765	4.4050	3.7531	3.2453	2.08 hours
lower 95% C.I.	8.1075	6.1866	4.9248	4.0647	3.4249	2.9292	
WT	8.5321	6.6136	5.2855	4.3806	3.6970	3.1590	15.75 second

- 50,000 simulation pathes for European and American options.

- 5,000 simulated paths for outer loop and 5,000 simulated paths for inner loop for VIX option.

Convergence on N

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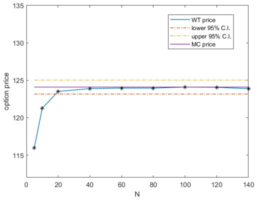
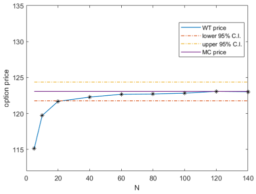
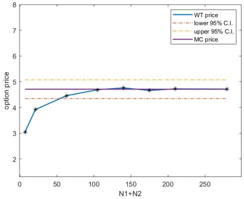


(e) European call



(f) American put



(g) VIX

Conclusion

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- Propose a novel willow tree method for non-parametric Heston-Duprie model.
- Replace the correlation decoupling with a joint probability distribution.
- Estimate the conditional expectation in the leverage function without simulations
- Provide the convergence rate of the willow tree method under the Heston-Duprie model.

Thank you!

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