Computational Recovery of the Time-Dependent Volatility of Volatility in a Heston Model

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The two-factor model introduces two sources of uncertainty by incorporating a stochastic variance:

$$\mathrm{d}x_t = \left(r_d - r_f - \frac{v_t}{2}\right) x_t \mathrm{d}t + \sqrt{v_t}\mathrm{d}W_t^1,$$
$$\mathrm{d}v_t = k(t)\big(\theta(t) - v_t\big)\mathrm{d}t + \xi(t)\sqrt{v_t}\mathrm{d}W_t^2.$$

where x_t is the log-spot price $x_t = \log S_t$, v_t is the instantaneous variance, $k(t)$ is the speed of mean reversion, $\theta(t)$ is the long-term mean of the variance, and $\xi(t)$ is the volatility of the variance.

The Wiener processes W_t^i are correlated with instantaneous correlation $\rho(t)$, i. e.

$$\mathbb{E}[\mathrm{d}W_t^1 \mathrm{d}W_t^2] = \rho(t)\mathrm{d}t.$$

Related Sources

Related Sources

- P. Carr, A. Itkin, D. Muravey, Semi-analytical pricing of barrier options in the time-dependent Heston model, arXiv:2202.06177 [q-fin.PR], (2022).
- A. Clevenhaus, C. Totzeck, M. Ehrhardt, A gradient based calibration method for the Heston model, Int. J. Comp. Math., (2024).
- S. Georgiev, L. Vulkov, Computational recovery of time-dependent volatility from integral observations in option pricing, J. Comput. Sci., 39, 101054, (2019).
- D. Guterding, W. Boenkost, The Heston stochastic volatility model with piecewise constant parameters – efficient calibration and pricing of window barrier options, J. Comput. Appl. Math., 343, 353-362, (2018).
- Y. Jin, J. Wang, S. Kim, Y. Heo, C. Yoo, Y. Kim, J. Kim, D. Jeong, Reconstruction of the time-dependent volatility function using the Black–Scholes model, Discr. Dyn. Nat. Soc., vol. 2018, ID 3093708, (2018).

Heston Model

We consider the two-dimensional general Heston equation for pricing European call option

$$\frac{\partial C}{\partial \tau} - \frac{v}{2}\frac{\partial^2 C}{\partial x^2} - \frac{\xi^2(\tau)}{2} v \frac{\partial^2 C}{\partial v^2} - \xi(\tau)\rho(\tau) v \frac{\partial^2 C}{\partial x \partial v} - \left(r_d - r_f - \frac{v}{2}\right)\frac{\partial C}{\partial x} - k(\tau)\big(\theta(\tau) - v\big)\frac{\partial C}{\partial v} + r_d C = 0 \quad (1)$$

with $\tau = T - t$ and the initial condition

$$C(x, v, 0) = \max\big(\exp(x) - K,\ 0\big).$$

Heston BC

In the solution to the direct and inverse problems, we use the Heston boundary conditions:

$$C(-\infty, v, \tau) = 0,$$

$$C(\infty, v, 0) = \exp(x) - K\exp(-r_d\tau),$$

$$\frac{\partial C}{\partial \tau}(x,0,\tau) - (r_d - r_f)\frac{\partial C}{\partial x}(x,0,\tau) - k(\tau)\theta(\tau)\frac{\partial C}{\partial v}(x,0,\tau) + r_d C(x,0,\tau) = 0,$$

$$C(x, \infty, 0) = \exp(x) - K\exp(-r_d\tau).$$

The question about well-posed boundary conditions would be regarded henceforward.

Implied Volatility of Volatility

Assume we know the option price C. Then we find that volatility $\xi(\tau)$, for which the theoretical result coincides with the observed quoted price on the market. This volatility is called *implied volatility*, i. e.

$$C^{\text{obs}} = C\big(x, t; K, T, r_d, r_f, \theta(\tau), k(\tau), \rho(\tau), \xi^{\text{imp}}(\tau)\big).$$

Discretization of (1)

Let $f : [x_{\min}, x_{\max}] \to \mathbb{R}$ and if $x_{\min} = x_0 < x_1 < \ldots < x_{I+1} = x_{\max}$ is the spatial grid, $h_i = x_i - x_{i-1}$, $H_i = h_i + h_{i+1}$, $(1 \leqslant i \leqslant I)$, then the first derivative $f'(x_i)$ could be approximated in the following ways:

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{h_i} \equiv \mathrm{D}^l f_i, \tag{2}$$

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{h_{i+1}} \equiv \mathrm{D}^r f_i \tag{3}$$

as well as the second derivative $f''(r_i)$:

$$f''(x_i) \approx \frac{2}{h_i H_i} f(x_{i-1}) - \frac{2}{h_i h_{i+1}} f(x_i) + \frac{2}{h_{i+1} H_i} f(x_{i+1}) \equiv \mathrm{D}^2 f_i. \tag{4}$$

Applying (2), (3) and (4) to (1), we have an upwind implicit scheme.

Rewriting (1) in terms of the gradient operator gives

$$\frac{\partial C}{\partial \tau} = \nabla \cdot (A \nabla C) + B^{\top} \cdot \nabla C - r_d C, \tag{5}$$

where

$$A = \frac{v}{2} \begin{bmatrix} 1 & \rho\xi \\ \rho\xi & \xi^2 \end{bmatrix}, \quad B = \begin{bmatrix} -\frac{v + \rho\xi}{2} + (r_d - r_f) \\ -\frac{\xi^2}{2} + k(\theta - v) \end{bmatrix}.$$

After multiplying (5) by a function $\phi \in H^1$, we obtain

$$\int_\Omega \frac{\partial C}{\partial \tau} \phi \mathrm{d}\Omega = \int_\Omega \nabla \cdot (A \nabla C) \phi \mathrm{d}\Omega - \int_\Omega B^{\top} \nabla C \phi \mathrm{d}\Omega - r_d \int_\Omega C \phi \mathrm{d}\Omega.$$

After applying the Green first identity to the diffusion term and choosing $\phi = C$, we get

$$\int_\Omega \frac{\partial C}{\partial \tau} C \mathrm{d}\Omega = \int_{\partial\Omega} C(A\nabla C)\cdot\vec{n}\mathrm{d}(\partial\Omega) - \int_\Omega \nabla C^\top A \nabla C \mathrm{d}\Omega + \int_\Omega C B^\top \nabla C \mathrm{d}\Omega - r_d \int_\Omega C^2 \mathrm{d}\Omega,$$

which, rewritten in terms of L^2 norms and accounting that $A \geqslant 0$, yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\|C\|^2 \leqslant \int_{\partial\Omega} C(A\nabla C)\cdot\vec{n}\mathrm{d}(\partial\Omega) + \int_\Omega C B^\top \nabla C \mathrm{d}\Omega - r_d\|C\|^2. \quad (6)$$

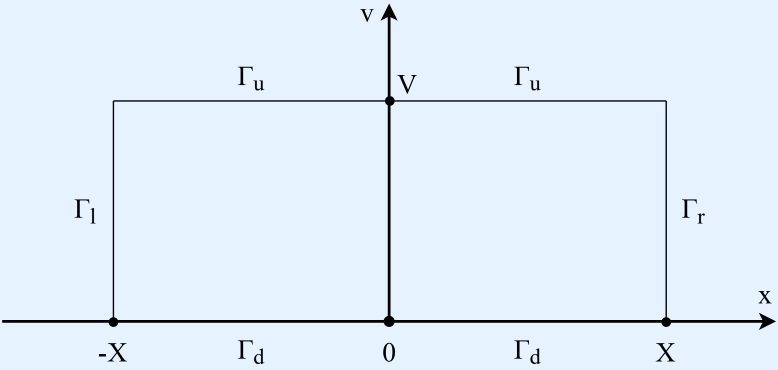
Theorem

If the integrals in (6) vanish or are negative, then the correspondin IBVP for eq. (1) is well-posed and the following estimate holds:

$$\|C(T)\| \leqslant \exp(bT)\|C(0)\|,$$

where $2b = \bar{K} - r_d$, $\bar{K} = \max_{\tau \in [0,T]} k(\tau)$.

Computational Domain



$$\Gamma_l = \{(x, v): \ x = -X, \ v \in (0, V)\},$$
$$\Gamma_r = \{(x, v): \ x = X, \ v \in (0, V)\},$$
$$\Gamma_u = \{(x, v): \ v = V, \ x \in (-X, X)\},$$
$$\Gamma_d = \{(x, v): \ v = 0, \ x \in (-X, X)\}.$$

Restricting x and v to a rectangular domain, truncated at $x_{\min} = -X$, $x_{\max} = X > 0$, $v_{\min} = 0$, $v_{\max} = V > 0$, forms four boundaries.

Diffusion terms

$$\int\limits_{\partial\Omega} C(A\nabla C)\cdot\vec{n}\mathrm{d}(\partial\Omega) = -\frac{1}{2}\int\limits_{0}^{V} vC\left(\rho(t)\xi(\tau)\frac{\partial C}{\partial v} + \frac{\partial C}{\partial x}\right)\mathrm{d}v\bigg|_{x=-X}$$

$$+\frac{1}{2}\int\limits_{0}^{V} vC\left(\rho(\tau)\xi(\tau)\frac{\partial C}{\partial v} + \frac{\partial C}{\partial x}\right)\mathrm{d}v\bigg|_{x=X}$$

$$+\frac{1}{2}\int\limits_{-X}^{X} vC\left(\xi^2(\tau)\frac{\partial C}{\partial v} + \rho(\tau)\xi(\tau)\frac{\partial C}{\partial x}\right)\mathrm{d}x\bigg|_{v=V}.$$

Analogously, the terms from the convection integral follows:

Convection terms

$$\int_\Omega CB^\top \nabla C \mathrm{d}\Omega = \underbrace{\frac{1}{2}\int_\Omega \left(k(\tau)\big(\theta(\tau) - v\big) - \frac{1}{2}\xi^2(\tau) \right) \frac{\partial C^2}{\partial v} \mathrm{d}\Omega}_{I_1}$$

$$+ \underbrace{\frac{1}{2}\int_\Omega (r_d - r_f)\frac{\partial C^2}{\partial x}\mathrm{d}\Omega}_{I_2} - \underbrace{\frac{1}{4}\int_\Omega \big(v + \rho(\tau)\xi(\tau)\big)\frac{\partial C^2}{\partial x}\mathrm{d}\Omega}_{I_3} .$$

Now, we will define the respective well-posed boundary conditions on the four boundaries.

After integrating by parts we obtain:

$$I_1 = \int_{-X}^{X}\int_{0}^{V}\left(k(\tau)\big(\theta(t)-v\big)-\frac{1}{2}\xi^2(\tau)\right)\frac{\partial C^2}{\partial v}\mathrm{d}v\mathrm{d}x$$

$$+\int_{-X}^{X}\left(k(\tau)\big(\theta(t)-V\big)-\frac{1}{2}\xi^2(\tau)\right)C^2(V,x,\tau)\mathrm{d}x$$

$$-\int_{-X}^{X}\left(k(\tau)\theta(\tau)-\frac{1}{2}\xi^2(\tau)\right)C^2(0,x,\tau)\mathrm{d}x+k(\tau)\iint_{\Omega}C^2\mathrm{d}\Omega.$$

Again, integrating by parts yields:

$$I_2 = (r_d - r_f) \int_0^V \big(C^2(v, X, \tau) - C^2(v, -X, \tau)\big) \mathrm{d}v$$

and

$$I_3 = \int_0^V \left(\big(v + \rho(\tau)\xi(\tau)\big) \int_{-X}^{X} \frac{\partial C^2}{\partial x} \mathrm{d}x \right) \mathrm{d}v =$$

$$\int_0^V \big(v + \rho(\tau)\xi(\tau)\big)\big(C^2(v, X, \tau) - C^2(v, -X, \tau)\big) \mathrm{d}v.$$

Convection terms

$$\int\limits_{\Omega} CB^{\top}\nabla C \mathrm{d}\Omega = \frac{1}{2}\int\limits_{-X}^{X}\left(k(\tau)\big(\theta(\tau)-V\big)-\frac{1}{2}\xi^2(\tau)\right)C^2(x,V,\tau)\mathrm{d}x$$

$$-\frac{1}{2}\int\limits_{-X}^{X}\left(k(\tau)\theta(\tau)-\frac{1}{2}\xi^2(\tau)\right)C^2(x,0,\tau)\mathrm{d}x+\frac{k(\tau)}{2}\iint\limits_{\Omega}C^2\mathrm{d}\Omega$$

$$+\frac{1}{2}\int\limits_{0}^{V}\left((r_d-r_f)-\frac{1}{2}\big(v+\rho(\tau)\xi(\tau)\big)\right)C^2(v,X,\tau)\mathrm{d}v$$

$$-\frac{1}{2}\int\limits_{0}^{V}\left((r_d-r_f)-\frac{1}{2}\big(v+\rho(t)\xi(\tau)\big)\right)C^2(v,-X,\tau)\mathrm{d}v.$$

Left boundary $\Gamma_l: \ x = -X$

$$-\frac{1}{2}\int_0^V \Bigg[v\left(\rho(\tau)\xi(\tau)\frac{\partial C}{\partial x}(-X,V,\tau) + \frac{\partial C}{\partial v}(-X,V,\tau)\right) + \left((r_d - r_f) - \frac{1}{2}\big(v + \rho(\tau)\xi(\tau)\big)\right) C^2(-X,V,\tau)\Bigg] \mathrm{d}v.$$

Down boundary $\Gamma_d: \ v = 0$

$$\frac{1}{2}\int_{-X}^{X}\left(k(\tau)\theta(\tau) - \frac{1}{2}\xi^2(\tau)\right) C^2(x,0,\tau)\mathrm{d}x.$$

Right boundary $\Gamma_r: \ x = X$

$$\frac{1}{2}\int_0^V \left(\rho(\tau)\xi(\tau)\frac{\partial C}{\partial x}(X,v,\tau) + \frac{\partial C}{\partial v}(X,v,\tau)\right)\mathrm{d}v$$

$$+\frac{1}{2}\int_0^V \left((r_d - r_f) - \frac{1}{2}\big(v + \rho(\tau)\xi(\tau)\big)\right) C^2(X,v,\tau)\mathrm{d}v.$$

Up boundary $\Gamma_u : \ v = V$

$$\frac{v}{2}\int\limits_{-X}^{X}\left(\xi^2(\tau)\frac{\partial C}{\partial x}(x,V,\tau)+\rho(\tau)\xi(\tau)\frac{\partial C}{\partial v}(x,V,\tau)\right)\mathrm{d}x$$

$$+\frac{1}{2}\int\limits_{-X}^{X}\left(k(\tau)\big(\theta(t)-V\big)-\frac{1}{2}\xi^2(\tau)\right)C^2(x,V,\tau)\mathrm{d}x.$$

Reconstruction Problem

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We have a set of *market* measurements $\{\omega_\beta^\alpha\}$, where ω_β^α is the quoted price of an option with maturity T_α, $\alpha = 1, \ldots, M_T$ and strikes K_β, $\beta = 1, \ldots, N$, assuming that $T_1 \leqslant \ldots \leqslant T_{M_T}$.

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We minimize the following

Cost Function

$$\Gamma_\alpha(\xi) = \frac{1}{N} \sum_{\beta=1}^{N} \left[c_\beta(\xi_\alpha(\tau_\alpha); K_\beta, T_\alpha) - \omega_\beta^\alpha \right]^2 \chi_\beta^\alpha,$$

$$\tau_\alpha \in (0, T_\alpha], \quad \alpha = 1, \ldots, M_T,$$

where $c_\beta(\xi_\alpha(\tau_\alpha); K_\beta, T_\alpha)$ is the numerical solution of (1) with strike K_β and expiry time T_α.

Step 1

Step 1

Step 1.1

We find μ_1 that minimizes the cost function

$$\Gamma_1(\xi) = \frac{1}{N}\sum_{\beta=1}^{N}\left[c_\beta(\xi_1(\tau_1); K_\beta, T_1) - \omega_\beta^1\right]^2 \chi_\beta^1,$$

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$$\tau_1 \in (0, T_1].$$

Step 1.2

We assume that the volvol function on $(0, T_1]$ is *constant*, defined as $\{\xi_1(\tau)\} = \mu_1$. Then we have

$$\xi(\tau) = \xi_1(\tau) \qquad \text{for } \tau \in [0, T_1].$$

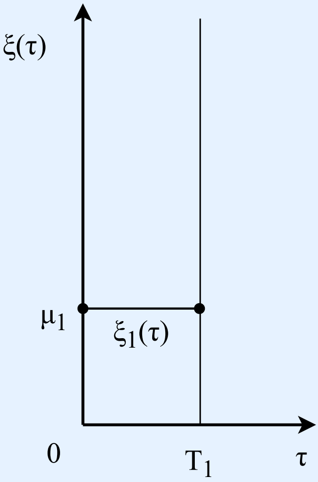

Figure: $\xi_1(\tau)$

Step 2

Step 2

Step 2.1

We find μ_2 that minimizes the cost function

$$\Gamma_2(\xi) = \frac{1}{N} \sum_{\beta=1}^{N} \left[c_\beta(\xi_2(\tau_2); K_\beta, T_2) - \omega_\beta^2 \right]^2 \chi_\beta^2,$$

$$\tau_2 \in (0, T_2].$$

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$$\tau_2 \in (0, T_2].$$

Step 2.2

We assume that the volvol function on $(0, T_1]$ is *linear*, defined as $\xi_2(\tau) = a\tau + b$. Then we have

$$\xi(\tau) = \xi_2(\tau) \qquad \text{for } \tau \in [0, T_2].$$

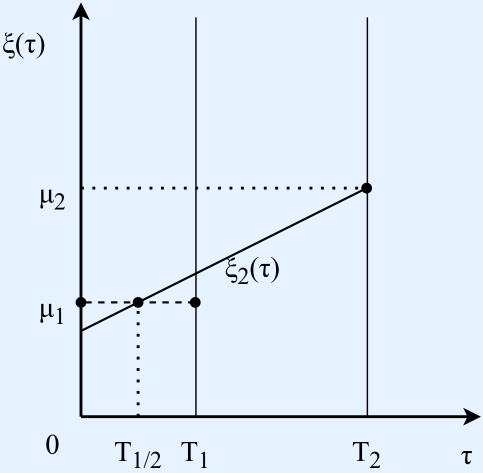

Figure: $\xi_2(\tau)$

Step 3

Step 3 is repeated from $\alpha = 3$ to $\alpha = M_T$.

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We find $\mu_\alpha := \{\xi_\alpha\}$ that minimizes the cost function

$$\Gamma_\alpha(\xi) = \frac{1}{N}\sum_{\beta=1}^{N}\left[c_\beta(\xi_\alpha(\tau_\alpha); K_\beta, T_\alpha) - \omega_\beta^\alpha\right]^2 \chi_\beta^\alpha, \qquad \tau_\alpha \in (0, T_\alpha].$$

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$$\tau_\alpha \in (0, T_\alpha].$$

Step 3.2

We define the *linear* volvol function $\xi_\alpha(\tau)$ on $[T_{\alpha-3/2}, T_\alpha]$ as

$$\xi_\alpha(\tau) = \frac{\mu_\alpha - \mu_{\alpha-1}}{T_\alpha - T_{\alpha-3/2}}(\tau - T_\alpha) + \mu_\alpha.$$

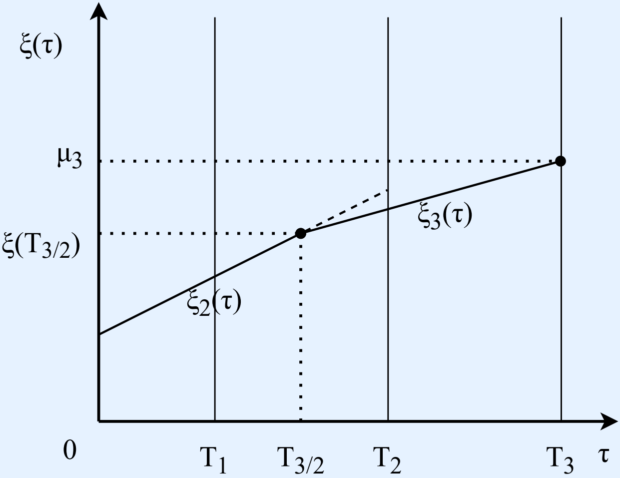

Figure: $\xi_3(\tau)$

Step 3

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If $\alpha = 2$, then the volvol function is *linear*:

$$\xi(\tau) = \xi_2(\tau) \qquad \text{for } \tau \in [0, T_2].$$

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If $\alpha \geqslant 3$, the volvol function is *piecewise linear*:

$$\xi(\tau) = \begin{cases} \xi_2(\tau) & \text{for } \tau \in [0, T_{3/2}], \\ \xi_j(\tau) & \text{for } \tau \in [T_{j-3/2}, T_{j-1/2}] \text{ for } 2 < j < \alpha, \\ \xi_\alpha(\tau) & \text{for } \tau \in [T_{\alpha-3/2}, T_\alpha], \end{cases}$$

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Finally, we arrive at the recovered volvol function $\xi(\tau)$ for $\tau \in (0, T_{M_T}]$.

For our synthetic data test we take

- $x_{\min} = -6$
- $x_{\max} = 6$
- $v_{\min} = 0$
- $v_{\max} = 10$
- $T = 5$ years
- $r_d = 0.05$
- $r_f = 0$
- $\theta(\tau) = 0.2$
- $k(\tau) = 2$
- $\rho(\tau) = -0.5$
- $\xi(\tau) = 0.005 + 0.004 \log(\tau + 1/3)$

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- $\xi(\tau) = 0.005 + 0.004 \log(\tau + 1/3)$

When solving the direct problem we take $\triangle\tau = 1/52$.

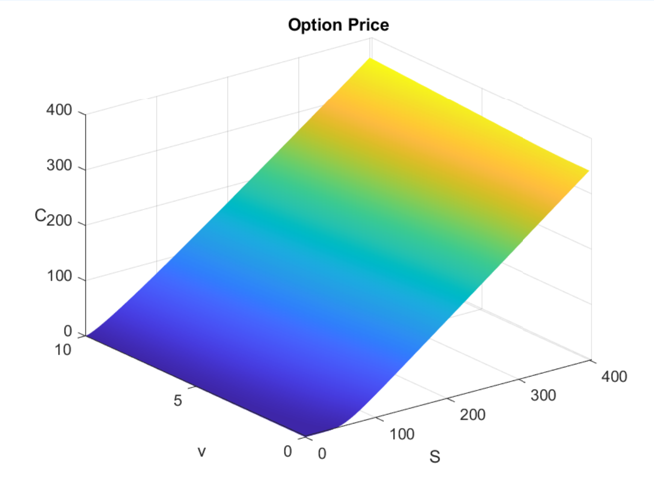

Figure: Option price

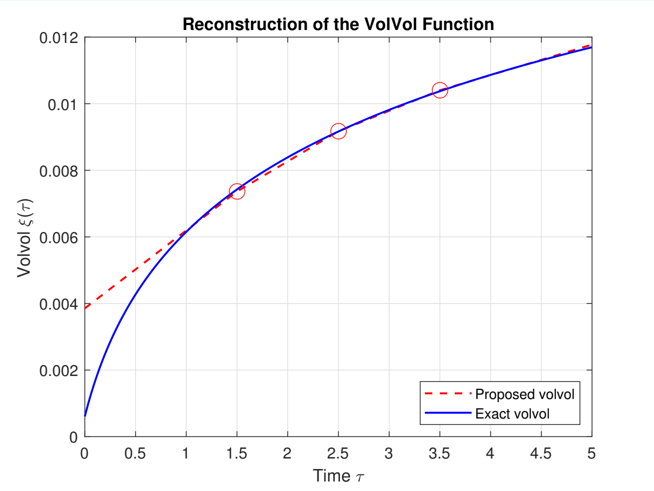

Figure: True and recovered volvol

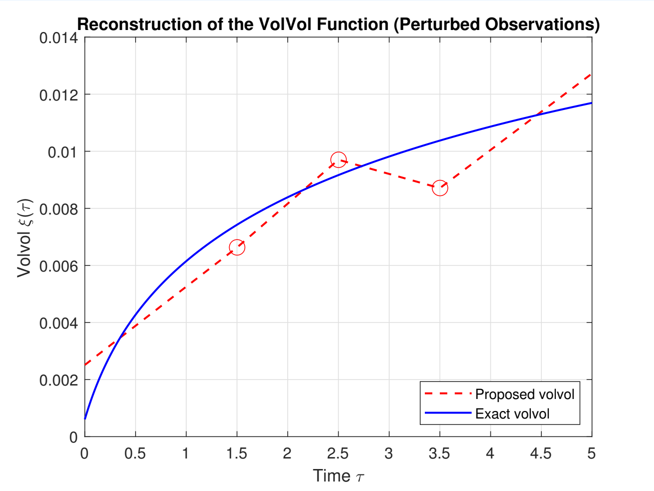

Figure: True and recovered volvol with perturbed observations

Implications

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The model suits the real market situation.

We adopt a *predictor-corrector* mechanism. At the first step, we assume the volatility is constant. Further, our algorithm builds a linear forward step, which corrects the volatility at half-backward time level. This is done for all the steps except the last one.

The reconstructed volvol function is *piecewise linear*.

The method does not require to invert a formula or an equation.

The algorithm is robust since we need to find only scalar parameters at each step.

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Future research:

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- American and exotic options

Thank you for your attention!