Affine Volterra processes with jumps

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joint work with
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Outline

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Affine Volterra process with jumps (one dimension)

A predictable process $\boldsymbol{X}$ with trajectories in $L^1_{\text{loc}}(\mathbb{R}_+)$ satisfying

$$\boldsymbol{X_t} = g_0(t) + \int_0^t \boldsymbol{K(t-s)}\,\mathrm{d}\boldsymbol{Z_s} = g_0(t) + (K * \mathrm{d}Z)_t, \quad \mathbb{P}\otimes \mathrm{d}t - \text{a.e.} \quad (1)$$

where $\boldsymbol{Z}$ is a semimartingale with (Lebesgue–) differential characteristics

$$(b(X),\ a(X),\ \eta(X, \mathrm{d}\xi))$$

- **Kernel:** $\boldsymbol{K} \in L^2_{\text{loc}}(\mathbb{R}_+)$ (e.g. $\boldsymbol{K(t) = t^{H-\frac{1}{2}}}$)
- **Initial input curve:** $g_0 \in L^1_{\text{loc}}(\mathbb{R}_+)$
- **Affine coefficients:**

$$b(x) = b_0 + xb_1, \quad a(x) = a_0 + xa_1, \quad \eta(x, \mathrm{d}\xi) = \nu_0(\mathrm{d}\xi) + x\nu_1(\mathrm{d}\xi),$$

with $\nu_k(\{0\}) = 0$ and $\int_{\mathbb{R}} |\xi|^2 \nu_k(\mathrm{d}\xi) < \infty$, $k = 0, 1$

Our goal

- Exponential–affine formulas for conditional Fourier–Laplace transform of X
- Extension of the results in [Abi Jaber, Larsson, Pulido ('19)] to **include jumps** (nontrivial...)

Our motivation

- Rough Hawkes Heston model [Bondi, Pulido, Scotti ('24)]: **joint SPX–VIX calibration** via Fourier–inversion techniques
- Semi–explicit transform formulas for (marked) **Hawkes processes**

Rough Hawkes Heston model [Bondi, Pulido, Scotti ('24)]

$$\mathrm{d}\boldsymbol{X}_t = -\left[\frac{1}{2} + \int_{\mathbb{R}_+} \left(e^{-\Lambda z} - 1 + \Lambda z\right) \nu(\mathrm{d}z)\right] \sigma_t^2 \,\mathrm{d}t + \sigma_t \left(\rho \mathrm{d}W_t + \sqrt{1-\rho^2}\mathrm{d}W_t^\perp\right)$$

$$-\Lambda \int_{\mathbb{R}_+} z\, \tilde{\mu}(\mathrm{d}t, \mathrm{d}z)$$

log return process

$$\boldsymbol{\sigma}_t^2 = g_0(t) + \int_0^t K(t-s)\mathrm{d}Z_s, \quad \mathbb{P} \otimes \mathrm{d}t$$

(spot variance)

$$\mathrm{d}Z_t = b\sigma_t^2 \mathrm{d}t + \sqrt{c}\sigma_t \mathrm{d}W_t + \int_{\mathbb{R}_+} z \tilde{\mu}(\mathrm{d}t, \mathrm{d}z)$$

- $b \in \mathbb{R}$, $c \geq 0$, $\tilde{\mu}$ compensated jump measure of $\boldsymbol{Z}$, with compensator $\boldsymbol{\sigma}_t^2 \mathrm{d}t \otimes \nu(\mathrm{d}z)$, where $\nu(\mathbb{R}_-) = 0$ and $\int_{\mathbb{R}_+} |z|^2 \nu(\mathrm{d}z) < \infty$
- $g_0(t) = \sigma_0^2 + \beta \int_0^t K(s)\,\mathrm{d}s$, $\beta, \sigma_0^2 \geq 0$ and $\Lambda > 0$

- **Jump times**: increasing sequence $(\tau_n)_n$ of stopping times
- **Jump sizes** (marks): i.i.d. sequence $(Y_n)_n$ of real square–integrable RVs distributed $\sim \theta(\mathrm{d}\xi)$

Marked Hawkes process N

A càdlàg self–exciting jump process $N_t = \sum_{t \geq \tau_n} Y_n$ with intensity

$$\lambda_t = \lambda_0 + \sum_{t > \tau_n} K(t - \tau_n) Y_n$$

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Then $$\lambda = \lambda_0 + (K * \mathrm{d}N), \quad \mathbb{P} \otimes \mathrm{d}t - \text{a.e.}$$

$\Longrightarrow$ **(1)** satisfied with: $X = \lambda$, $Z = N$, $g_0 = \lambda_0$, $b_1 = \int_{\mathbb{R}} \xi \theta(\mathrm{d}\xi)$, $\nu_1(\mathrm{d}\xi) = \theta(\mathrm{d}\xi)$,
$a_0 = a_1 = b_0 = \nu_0 = 0$

Challenge

If K explodes at $0 \Longrightarrow \lambda$ jumps to infinity at τ_n !

- **[Abi Jaber, Cuchiero, Larsson, Pulido ('21)]**: existence of L^p_{loc}, $p \geq 2$, solutions if K satisfies, for some $\eta \in (0,1)$,

$$\int_0^T |K(t)|^p t^{-p\eta} \mathrm{d}t + \int_0^T \int_0^T \frac{|K(t) - K(s)|^p}{|t-s|^{1+p\eta}} \mathrm{d}s \, \mathrm{d}t \leq \underbrace{c_K(T)}_{\text{locally bounded fcn.}}$$

Example with $p = 2$: fractional kernel $K(t) = t^{H-\frac{1}{2}}$, $H \in (0, \frac{1}{2}]$

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- **[Abi Jaber ('21)]**: existence of **nonnegative** L^1_{loc} solutions to **(1)** via

$$Y_t = \int_0^t g_0(s)\, \mathrm{d}s + \int_0^t K(t-s) Z_s \, \mathrm{d}s, \quad \text{char}\,(Z) = (b_1 Y, a_1 Y, Y\nu_1(\mathrm{d}\xi))$$

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- [Cuchiero, Teichmann ('20)]: **Existence + uniqueness marginals** via solutions of measure–valued SPDEs

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Fourier-Laplace transform formula of X [Bondi, Livieri, Pulido ('22)]

Under suitable conditions on K and f

$$\mathbb{E}_t\left[\mathrm{e}^{\int_t^T f(T-s)X_s\,\mathrm{d}s}\right] = \exp\left(\phi(T-t) + \int_t^T \mathcal{R}(T-s,\psi(T-s))g_t(s)\mathrm{d}s\right)$$

where

$$g_t(s) = g_0(s) + \int_0^t K(s-r)\,\mathrm{d}Z_r$$

$$\mathcal{R}(s,z) = f(s) + \frac{1}{2}z^2 a_1 + z b_1 + \int_{\mathbb{R}} \left(\mathrm{e}^{z\xi} - 1 - z\xi\right)\nu_1(\mathrm{d}\xi)$$

$$\psi(t) = \int_0^t K(t-s)\mathcal{R}(s,\psi(s))\,\mathrm{d}s$$

$$\phi(t) = \int_0^t \left(\psi(s)b_0 + \frac{1}{2}\psi(s)^2 a_0 + \int_{\mathbb{R}} \left(\mathrm{e}^{\psi(s)\xi} - 1 - \psi(s)\xi\right)\nu_0(\mathrm{d}\xi)\right)\mathrm{d}s$$

Idea of the proof

- Let $V_t^T = \phi(T-t) + \int_0^t f(T-s)X_s \mathrm{d}s + \int_t^T \mathcal{R}\left(T-s, \psi\left(T-s\right)\right) g_t\left(s\right) \mathrm{d}s$
- **Stochastic Fubini + Itô** show that $\exp\left\{V^T\right\}$ is a local martingale

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- **Stochastic Fubini + Itô** show that $\exp\left\{V^T\right\}$ is a local martingale
- Denoting by $\boldsymbol{L}$ the resolvent of the first kind of K (i.e. $\boldsymbol{K * L = 1}$)

$$g_t\left(s\right) = g_0\left(s\right) + K\left(s-t\right) Z_t + \left(\left(\left(\Delta_{s-t}K\right)' * \left(X - g_0\right)\right) * \boldsymbol{L}\right)_t$$

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- Plug the new expression of $g_t(\cdot)$ in the definition of V_t^T to get

$$V_t^T = \phi(T-t) + \int_0^t f(T-s)X_s \mathrm{d}s + \int_0^{T-t} g_0(T-s)\, \mathcal{R}(s, \psi(s))\, \mathrm{d}s + \psi(T-t)\, L(\{0\})\, (X - g_0)(t) - \left(\mathrm{d}\tilde{\Pi}_{T-t} * g_0\right)_t + \left(\mathrm{d}\tilde{\Pi}_{T-t} * X\right)_t \quad (2)$$

where $\tilde{\Pi}_h$, $h > 0$, is an AC function depending on ψ and L

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- Plug the new expression of $g_t(\cdot)$ in the definition of V_t^T to get

$$\begin{aligned} V_t^T = \phi(T-t) &+ \int_0^t f(T-s)X_s \mathrm{d}s + \int_0^{T-t} g_0(T-s)\, \mathcal{R}(s, \psi(s))\, \mathrm{d}s \\ &+ \psi(T-t)\, L(\{0\})\, (X - g_0)(t) - \left(\mathrm{d}\tilde{\Pi}_{T-t} * g_0\right)_t + \left(\mathrm{d}\tilde{\Pi}_{T-t} * X\right)_t \end{aligned} \tag{2}$$

  where $\tilde{\Pi}_h$, $h > 0$, is an AC function depending on ψ and L

- Use (2) and **comparison results for Riccati–Volterra equations** to show that $\exp\left\{V^T\right\}$ is a **martingale** ■

Fourier–Laplace transform formula Z [Bondi, Livieri, Pulido ('22)]

For every $T > 0$, under suitable conditions on K

$$\mathbb{E}_t\left[e^{uZ_T}\right] = \exp\left(uZ_t + \phi(T-t) + \int_t^T \mathcal{R}(T-s, \psi(T-s)) g_t(s)\,\mathrm{d}s\right)$$

whenever the **local** martingale in the RHS is **true**.

Here $\psi(t) = u + \int_0^t K(t-s)\mathcal{R}(s, \psi(s))\,\mathrm{d}s$, with

$$\mathcal{R}(s,z) = zb_1 + \frac{1}{2}z^2 a_1 + \int_{\mathbb{R}} \left(e^{z\xi} - 1 - z\xi\right)\nu_1(\mathrm{d}\xi)$$

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Example: Marked Hawkes process N

$$\mathbb{E}_t\left[\exp\{uN_T\}\right] = \exp\left(uN_t + \int_t^T \left(\int_{\mathbb{R}} (\mathrm{e}^{\psi(T-s)\xi} - 1)\theta(\mathrm{d}\xi)\right) g_t(s)\,\mathrm{d}s\right)$$

where $g_t(s) = \lambda_0 + \int_0^t K(s-r)\,\mathrm{d}N_r$ and

$$\psi(t) = u + \int_0^t K(t-s)\left(\int_{\mathbb{R}} (\mathrm{e}^{\psi(s)\xi} - 1)\theta(\mathrm{d}\xi)\right)\mathrm{d}s$$

Related literature: [Gatheral, Keller-Ressel ('19)]

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Thank you!

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