Control and optimal stopping Mean-field games: a linear programming approch

Roxana Dumitrescu[1]

Based on joint works with

René Aïd[2] Geraldine Bouveret[3] Marcos Leutscher[4] Peter Tankov [4]

[1]King's College London

[2]Paris-Dauphine University

[3]RIMM Sustainability

[4]ENSAE, Institut Polytechnique de Paris

The International Conference on Computational Finance ICCF 2024

Outline

Mean-field games: an introduction

Foundations and applications

- Introduced by Lasry and Lions (2006, 2007) and Huang, Caines, Malhamé (2006) using PDE tools to describe large-population games with symmetric interactions in a tractable way
- Numerous applications in economics, finance, engineering, epidemiology etc.

  Systemic risk (e.g. Carmona, Fouque, Sun '15, '18), price impact and optimal execution (e.g. Cardaliaguet-Lehalle '16, Cartea-Jaimungal-Penalva '18), models for oil production (Guéant-Lasry-Lions '10, Chan-Sircar '17), cryptocurrencies and bitcoin mining (Bertucci, Bertucci, Lasry and Lions '20), models for energy markets, environment economics etc..

For a very recent review of applications: Carmona (2020).

Mean-field games: an introduction

N-players game formulation

- Each player controls its state $X_t^i \in \mathbb{R}^d$ by taking an action $\alpha_t^i \in A \subset \mathbb{R}^k$:

$$dX_t^i = b(t, X_t^i, \bar{\mu}_{X_t^{-i}}^{N-1}, \alpha_t^i)dt + \sigma(t, X_t^i, \bar{\mu}_{X_t^{-i}}^{N-1}, \alpha_t^i)dW_t^i,$$

W^i are independent and $\bar{\mu}_{X_t^{-i}}^{N-1}$ is the empirical distribution of other players.

Mean-field games: an introduction

N-players game formulation

- Each player controls its state $X_t^i \in \mathbb{R}^d$ by taking an action $\alpha_t^i \in A \subset \mathbb{R}^k$:

$$dX_t^i = b(t, X_t^i, \bar{\mu}^{N-1}_{X_t^{-i}}, \alpha_t^i)dt + \sigma(t, X_t^i, \bar{\mu}^{N-1}_{X_t^{-i}}, \alpha_t^i)dW_t^i,$$

  W^i are independent and $\bar{\mu}^{N-1}_{X_t^{-i}}$ is the empirical distribution of other players.

- Each player minimises the cost

$$J^i(\boldsymbol{\alpha}) = \mathbb{E}\left[\int_0^T f(t, X_t^i, \bar{\mu}^{N-1}_{X_t^{-i}}, \alpha_t^i)dt + g(X_T^i, \bar{\mu}^{N-1}_{X_T^{-i}})\right],$$

- We look for a Nash equilibrium $\hat{\boldsymbol{\alpha}}$: $\forall i, \ \forall \alpha^i, \ J^i(\hat{\boldsymbol{\alpha}}) \leq J^i(\alpha^i, \hat{\boldsymbol{\alpha}}^{-i})$.

Mean-field games: an introduction

Towards a mean-field game

When the number of agents is large, it is natural to consider the following limiting version of the game:

- The *representative player* controls its state X^α depending on the deterministic flow $(\mu_t)_{0\leq t\leq T}$, which corresponds to the distribution of states of all players:

$$dX_t^\alpha = b(t, X_t^\alpha, \mu_t, \alpha_t)dt + \sigma(t, X_t^\alpha, \mu_t, \alpha_t)dW_t.$$

- The aim of the player is to minimize the cost

$$\inf_{\alpha\in A} J^\mu(\alpha), \quad J^\mu(\alpha) = \mathbb{E}\left[\int_0^T f(t, X_t^\alpha, \mu_t, \alpha_t)dt + g(X_T^\alpha, \mu_T)\right] \quad (*)$$

Mean-field games: an introduction

Towards a mean-field game

When the number of agents is large, it is natural to consider the following limiting version of the game:

- The *representative player* controls its state X^α depending on the deterministic flow $(\mu_t)_{0\le t\le T}$, which corresponds to the distribution of states of all players:

$$dX_t^\alpha = b(t, X_t^\alpha, \mu_t, \alpha_t)dt + \sigma(t, X_t^\alpha, \mu_t, \alpha_t)dW_t.$$

- The aim of the player is to minimize the cost

$$\inf_{\alpha\in A} J^\mu(\alpha), \quad J^\mu(\alpha) = \mathbb{E}\left[\int_0^T f(t, X_t^\alpha, \mu_t, \alpha_t)dt + g(X_T^\alpha, \mu_T)\right] \quad (*)$$

- A mean-field equilibrium is a flow $(\mu_t)_{0\le t\le T}$ such that $\mathcal{L}(\hat{X}_t^\mu) = \mu_t$, $t \in [0, T]$, where $\hat{X}^\mu$ is the solution to $(*)$.

Mean-field games: an introduction

Approaches

- **PDE approach**: developed by Lasry and Lions (2006, 2007) and Huang, Malhamé and Caines (2006) $\rightarrow$ coupled system of partial differential equations: *Hamilton-Jacobi-Bellman (backward)* and *Fokker-Planck-Kolmogorov (forward)*.
- **FBSDE approach**: introduced by Carmona and Delarue (2012) $\rightarrow$ coupled *forward-backward stochastic differential equations* with coefficients which depend on the law of the solution.
- **Compactification methods**: Allow to solve the problem under mild assumptions by relaxing the concept of equilibrium.
  $\rightarrow$ Controlled martingale problem (Lacker (2015)).

Mean-field games: an introduction

PDE approach

- The value function associated to the stochastic control problem is characterized as the solution to a HJB equation

$$\partial_t V + \max_{\alpha} \left\{ f(t,x,\mu_t,\alpha) + b(t,x,\mu_t,\alpha)\partial_x V + \frac{1}{2}\sigma^2(t,x,\mu_t,\alpha)\partial^2_{xx} V \right\} =$$

with the terminal condition $V(T,x) = g(x,\mu_T)$.

- The flow of densities solves the Fokker-Planck equation

$$\partial_t \mu_t - \frac{1}{2}\partial^2_{xx}(\sigma^2(t,x,\mu_t,\hat{\alpha}_t)\mu_t) + \partial_x(b(t,x,\mu_t,\hat{\alpha}_t)\mu_t) = 0,$$

with the initial condition $\mu_0 = \delta_{X_0}$, where $\hat{\alpha}$ is the optimal feedback control.

$\Rightarrow$ A coupled system of a Hamilton-Jacobi-Bellman PDE (backward) and a Fokker-Planck PDE (forward)

Outline

Optimal stopping MFG

State of the art

In optimal stopping mean-field games (aka MFG of timing), the strategy of each agent is a stopping time.

- Nutz (2017): bank run model with common noise, interaction through proportion of stopped players;
- Carmona, Delarue and Lacker (2017): a general timing game with common noise, interaction through proportion of stopped players. Pre-emption game.
- Bertucci (2017): Markovian state of each agent; no common noise, interaction through density of states of players still in the game, analytic approach (obstacle problem), existence of mixed equilibria. War of attrition.

Optimal stopping MFG

New approach: Linear programming formulation of MFG

- A compactification technique, inspired by works on LP formulation of stochastic control (Stockbridge '90, Cho and Stockbridge '02).
- Particularly suitable for MFG with optimal stopping and control with absorption: the lack of regularity of the flow μ_t makes it difficult to use the analytic approach.

Optimal stopping MFG

Rationale and advantages of LP formulation

- Instead of iterating back and forth between the value function of the single agent and the population dynamics, the problem is formulated exclusively in terms of the population measure flow, which is the main object of interest.
- The condition that the measure flow is the flow of marginal laws of a stochastic process gives a linear constraint on the measure flow.
- This formulation simplifies both the theoretical analysis of the problem (existence of equilibrium is established under weaker assumptions) and the numerical computation of solutions.
- Equivalence to *strong* formulations may be shown under appropriate assumptions.

Optimal stopping MFG

We develop theory and applications of the LP approach to MFG in a series of papers:

- Bouveret, Dumitrescu, and Tankov “Mean-field games of optimal stopping: a relaxed solution approach.” SIAM J. Con. Optim. 58.4 (2020).
- Aïd, Dumitrescu, and Tankov, “The entry and exit game in the electricity markets: a mean-field game approach." Journal of Dynamics and Games 8.4 (2021).
- Bouveret, Dumitrescu, and Tankov, “Technological Change in Water Use: A Mean-Field Game Approach to Optimal Investment Timing.” Operations Research Perspectives 9 (2022).
- Dumitrescu, Leutscher, and Tankov, “Control and optimal stopping Mean Field Games: a linear programming approach.” Electronic Journal of Probability 26 (2021).
- Dumitrescu, Leutscher, and Tankov, “Linear Programming Fictitious Play algorithm for Mean Field Games with optimal stopping and absorption”, to appear in ESAIM:Mathematical Modeling and Numerical Analysis
- Dumitrescu, Leutscher, and Tankov, “Energy transition under scenario uncertainty: a mean-field game approach.” arXiv:2210.03554 (2022).

Optimal stopping MFG

N-players game problem

- Consider N agents X^i, $i = 1, \dots, N$ with dynamics

$$dX_t^i = b(t, X_t^i)dt + \sigma(t, X_t^i)dW_t^i, \quad X_0^i \in \mathcal{O},$$

where W^i, $i = 1, \dots, N$ are independent.

In the talk, to simplify notation, we assume either $\mathcal{O} = \mathbb{R}$ or $\mathcal{O} \subset \mathbb{R}$ with unattainable boundary; $\mathbb{R}^n$ and absorbing boundary can also be considered.

Optimal stopping MFG

N-players game problem

- Each agent aims to solve the following *optimal stopping problem*:

$$\sup_{\tau} \mathbb{E}\left[\int_0^{\tau} f\left(t, X_t^i, m_t^{N-1}\right) dt + g\left(\tau, X_\tau^k, \mu^{N-1}\right)\right],$$

where

$$m_t^{N-1}(dx) = \frac{1}{N-1} \sum_{k=1;k\neq i}^{N-1} \delta_{X_t^k}(dx) \mathbf{1}_{t \leq \tau^k},$$

and

$$\mu^{N-1}(dt, dx) = \frac{1}{N-1} \sum_{k=1;k\neq i}^{N-1} \delta_{(\tau^k, X_\tau^k)}(dt, dx),$$

with τ^k is the stopping time chosen by the player k.

- Look for Nash equilibria.

Optimal stopping MFG

MFG formulation

As $N \to \infty$, we expect m^N converge to deterministic limit m.

- State process of the *representative agent*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)\, dW_t,$$

- The optimal stopping problem for the agent takes the form

$$\sup_{\tau} \quad \mathbb{E}\left[\int_0^{\tau} f\,(t, X_t, m_t)\, dt + g(\tau, X_\tau, \mu)\right]. \qquad (1)$$

Optimal stopping MFG

- Given the solution $\tau^{m,\mu}$ of the problem (1) for the agent facing a mean-field $((m_t)_{t\in[0,T]},\mu)$, find $((m_t)_{t\in[0,T]},\mu)$ such that

$$m_t(B) = \mathbb{P}\left[X_t \in B, t < \tau^{\mu,m}\right], B \in \mathcal{B}(\mathcal{O}), t \in [0,T]. \qquad (2)$$

and

$$\mu = \mathcal{L}\left(\tau^{\mu,m}, X_{\tau^{\mu,m}}\right). \qquad (3)$$

Solution of the optimal stopping MFG: fixed point of (2) – (3).

Linear programming approach for optimal stopping

Single agent problem

- We start with the **single agent problem** (no mean-field terms here):

$$
\begin{aligned}
\sup_{\tau \in \mathcal{T}} \quad & \mathbb{E}\left[\int_0^\tau f\,(t, X_t)\,dt + g(\tau, X_\tau)\right], \\
\text{s.t.} \quad & dX_t = b\,(t, X_t)\,dt + \sigma\,(t, X_t)\,dW_t, \\
& X_0 \sim m_0^\star.
\end{aligned}
$$

Linear programming approach for optimal stopping

- For any $\tau \in \mathcal{T}$, define the **flow of subprobability measures** m^τ and **the probability measure** μ^τ by

$$m_t^\tau(B) = \mathbb{P}\left(X_t \in B, t < \tau\right), \quad B \in \mathcal{B}(\mathcal{O}), \quad t \in [0, T],$$
$$\mu^\tau(C) = \mathbb{P}((\tau, X_\tau) \in C), \quad C \in \mathcal{B}([0, T] \times \bar{\mathcal{O}}).$$

- We can *rewrite the expected reward*

$$\mathbb{E}\left[\int_0^\tau f\left(t, X_t\right) dt + g\left(\tau, X_\tau\right)\right] = \int_0^T \int_{\mathcal{O}} f(t,x) m_t^\tau(dx) dt + \int_{[0,T]\times\bar{\mathcal{O}}} g(t,x)\mu^\tau(dt, dx).$$

Linear programming approach for optimal stopping

- We apply Itô's formula to $u \in C_b^{1,2}([0, T] \times \mathbb{R})$ up to time τ and we get

$$u(\tau, X_\tau) = u(0, X_0) + \int_0^\tau (\partial_t u + \mathcal{L}u)(t, X_t)dt + \int_0^\tau (\sigma \partial_x u)(t, X_t)dW_t,$$

where

$$\mathcal{L}u(t,x) = b(t,x)\partial_x u(t,x) + \frac{\sigma^2}{2}(t,x)\partial_{xx} u(t,x).$$

Linear programming approach for optimal stopping

- Taking the expectation in the above expression, we get

$$\int_{[0,T]\times\bar{\mathcal{O}}} u(t,x)\mu^\tau(dt,dx) = \int_{\mathbb{R}} u(0,x)m_0^*(dx) + \int_0^T \int_{\mathcal{O}} (\partial_t u + \mathcal{L}u)(t,x)m_t^\tau(dx)dt.$$

$\rightarrow$ The set of tuples (μ^τ, m^τ), $\tau \in \mathcal{T}$ is *included* in the set:

Definition

Let $\mathcal{R}$ be the set of (μ, m) such that for all $u \in C_b^{1,2}([0,T]\times\mathcal{O})$

$$\int_{[0,T]\times\bar{\mathcal{O}}} u(t,x)\mu(dt,dx) = \int_{\mathcal{O}} u(0,x)m_0^*(dx) + \int_0^T \int_{\mathcal{O}} (\partial_t u + \mathcal{L}u)(t,x)m_t(dx)dt.$$

Linear programming approach for optimal stopping

The **linear programming formulation** consists in solving the problem

$$V^{LP} := \sup_{(\mu,m)\in\mathcal{R}} \int_0^T \int_{\mathcal{O}} f(t,x) m_t(dx) dt + \int_{[0,T]\times\bar{\mathcal{O}}} g(t,x)\mu(dt,dx).$$

The initial problem is embedded in this one.

Linear programming approach for optimal stopping

Existence result

Assume that b, σ are measurable and Lipschitz in x, f is jointly measurable and continuous in x for all t, g is continuous with respect to (t, x) and f, g satisfy

$$|f(t,x)| \leq c(1+|x|^2)| \quad \text{and } |g(t,x)| \leq c(1+|x|^2)|$$

for some $c > 0$.

Theorem (*Existence of a solution for the LP problem*)

There exists a solution to the linear programming problem for the single agent, i.e. there exists $(\mu^\star, m^\star) \in \mathcal{R}$ such that

$$V^{LP} = \int_0^T \int_{\mathcal{O}} f(t,x) m_t^\star(dx)dt + \int_{[0,T]\times\bar{\mathcal{O}}} g(t,x)\mu^\star(dt,dx)$$

.

MFG linear programming formulation

- We denote by V_2 the set of (identified t-a.e.) *subprobability measure flows* $m = (m_t)_{t\in[0,T]} \subset \mathcal{P}^{sub}(\bar{\mathcal{O}})$ such that $t \mapsto m_t(B)$ is measurable for each $B \in \mathcal{B}(\mathcal{O})$, m_t is finite and $\int_0^T \int_{\bar{\mathcal{O}}} |x|^2 m_t(dx)dt < \infty$.
- We endow V_2 with the *topology of weak convergence* of the associated measures $m_t(dx)dt$, that is, we say that the sequence $(m_t^n)_{t\in[0,T]} \subset V_2$ converges to $(m_t)_{t\in[0,T]} \in V_2$ if for all continuous functions φ with quadratic growth,

$$\int_0^T \int_{\mathcal{O}} \varphi(t,x) m_t^n(dx)dt \quad \underset{n\to\infty}{\longrightarrow} \quad \int_0^T \int_{\mathcal{O}} \varphi(t,x) m_t(dx)dt.$$

- Let $\mathcal{P}_2([0,T] \times \bar{\mathcal{O}})$ be endowed with the *topology of weak convergence*.

MFG linear programming formulation

Fix a pair $(\bar{\mu}, \bar{m}) \in \mathcal{P}_2([0, T] \times \bar{\mathcal{O}}) \times V_2$.

- Let $\Gamma[\bar{\mu}, \bar{m}] : \mathcal{P}_2([0, T] \times \bar{\mathcal{O}}) \times V_2 \mapsto \mathbb{R}$ be defined by

$$\Gamma[\bar{\mu}, \bar{m}](\mu, m) = \int_0^T \int_{\bar{\mathcal{O}}} f(t, x, \bar{m}_t) m_t(dx) dt + \int_{[0,T]\times\bar{\mathcal{O}}} g(t, x, \bar{\mu}) \mu(dt, dx).$$

MFG linear programming formulation

- We say that $(\mu^\star, m^\star) \in \mathcal{P}_2([0,T] \times \bar{\mathcal{O}}) \times V_2$ is an *LP MFG Nash equilibrium* if $(\mu^\star, m^\star) \in \mathcal{R}$ and for all $(\mu, m) \in \mathcal{R}$,

$$\Gamma[\mu^\star, m^\star](\mu, m) \leq \Gamma[\mu^\star, m^\star](\mu^\star, m^\star).$$

The real number $\Gamma[\mu^\star, m^\star](\mu^\star, m^\star)$ is called a *Nash value*.

Existence of LP MFG equilibria

Main assumptions

- The coefficients $b : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $\sigma : [0, T] \times \mathbb{R} \to \mathbb{R}_+$ are jointly measurable, Lipschitz in x uniformly on t.
- The function $(t, x, m) \mapsto f(t, x, m)$ is jointly measurable and continuous in (x, m) for each t. The function g is jointly continuous. Moreover, there exists a constant $c_2 > 0$ such that for all (t, x, m, μ)

$$|f(t, x, m)| \leq c_2 \left(1 + |x|^2 + \int_{\bar{\mathcal{O}}} |z|^2 m(dz)\right),$$

$$|g(t, x, \mu)| \leq \left(1 + |x|^2 + \int_{\bar{\mathcal{O}}} |z|^2 \mu(ds, dz)\right).$$

Existence of LP MFG equilibria

- Define $\Theta : \mathcal{R} \to 2^{\mathcal{R}}$ as

$$\Theta(\bar{\mu}, \bar{m}) = \arg\max_{(\mu, m) \in \mathcal{R}} \Gamma[\bar{\mu}, \bar{m}](\mu, m).$$

$\Rightarrow$ the set of Nash equilibria coincides with the set of fixed points of the set-valued mapping Θ.

Theorem

The set of LP MFG Nash equilibria is compact and nonempty.

The proof is based on Kakutani-Fan-Glicksberg Theorem.

Uniqueness of the Nash value

Suppose also that f and g take the following form

$$f(t,x,m) = f_1(t,x) f_2\left(t, \int_{\mathbb{R}} f_1(t,y) m(dy)\right) + f_3(t,x)$$

$$g(t,x,\mu) = g_1(t,x) g_2\left(\int_{[0,T]\times\mathbb{R}} g_1(s,y)\mu(ds,dy)\right) + g_3(t,x),$$

where f_1, f_2, f_3, g_1, g_2, g_3 are bounded and measurable, f_2 is non-increasing in the second argument and g_2 is non-increasing.

Uniqueness of the Nash value

Let (μ^1, m^1) and (μ^2, m^2) be two LP Nash equilibria. Then,

$$f_2\left(t, \int_{\mathbb{R}} f_1(t, y, u) m_t^1(dy, du)\right) = f_2\left(t, \int_{\mathbb{R}} f_1(t, y, u) m_t^2(dy, du)\right),$$

almost everywhere on $[0, T]$, and

$$g_2\left(\int_{[0,T]\times\mathbb{R}} g_1(s, y)\mu^1(ds, dy)\right) = g_2\left(\int_{[0,T]\times\mathbb{R}} g_1(s, y)\mu^2(ds, dy)\right).$$

In particular they lead to the same Nash value, that is

$$\Gamma[\mu^1, m^1](\mu^1, m^1) = \Gamma[\mu^2, m^2](\mu^2, m^2).$$

Link with the strong formulation and mixed solutions

Link with the strong formulation Assume in particular that (i) σ is uniformly elliptic and (ii) the domain $\mathcal{O}$ is an open bounded domain.

Let $(\mu^\star, m^\star)$ be an LP Nash equilibrium. Consider the value function given by

$$v^\star(t,x) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\int_t^{\tau \wedge \tau_{\mathcal{O}}^{t,x}} f\left(s, X_s^{t,x}, m_s^\star\right) ds + g\left(\tau \wedge \tau_{\mathcal{O}}^{t,x}, X_{\tau \wedge \tau_{\mathcal{O}}^{t,x}}^{t,x}, \mu^\star\right)\right], \tag{4}$$

where $(t,x) \in [0,T] \times \mathbb{R}$, $\tau_{\mathcal{O}}^{t,x,m^\star,\alpha} := \inf\left\{s \geq t : X_s^{t,x} \notin \mathcal{O}\right\}$.

Link with the strong formulation and mixed solutions

Link with the strong formulation: We have

$$\int_{\mathcal{O}} v^{\star}(0,x)m_0^{*}(dx) = \Gamma[\mu^{\star}, m^{\star}](\mu^{\star}, m^{\star}).$$

Link with the strong formulation and mixed solutions

Link with mixed solutions We assume here for simplicity $g = 0$. We obtain that $(v^\star, m^\star)$ is a solution of the coupled system of equations

(a)
$$\int_{\mathcal{S}} f(t, x, m_t^\star) m_t^\star(dx)dt = 0,$$
with $\mathcal{S}^\star := \{(t,x) \in [0, T] \times \mathcal{O} : v^\star(t,x) = 0\}$.

(b) For all C^∞ functions ϕ such that $\text{supp}(\phi) \subset \mathcal{C}^\star$, the following holds
$$\int_{\mathcal{O}} \phi(0, x) m_0^*(dx) + \int_0^T \int_{\mathcal{O}} \left(\frac{\partial \phi}{\partial t} + \mathcal{L}\phi \right)(t, x, m_t^\star) m_t^\star(dx)dt = 0,$$
where $\mathcal{C}^\star := ([0, T] \times \mathcal{O}) \setminus \mathcal{S}^\star$.

(c)
$$\min(v^\star(t,x), -\frac{\partial}{\partial t} v^\star(t,x) - \mathcal{L}v^\star(t,x) - f(t,x,m^\star)) = 0,$$
for $(t,x) \in [0, T] \times \mathcal{O}$ and with terminal and boundary conditions $v^\star(T,x) = 0$; $v^\star(t,x) = 0, t \in [0, T], x \in \partial O$.

Numerical algorithm

State of the art

- Several numerical algorithms have been proposed in the literature in the case of **regular control (without absorption)** , using analytic and probabilistic approaches (e.g. Achdou, Guéant, Laurière, Chassagneux, Crisan, Delarue). Another method, based on the fictitious play algorithm (*learning procedure*) has been introduced by Cardaliaguet-Hadikhanloo in the context of MFG of controls.
- **Very few algorithms** in the case of **MFG of optimal stopping**: Bouveret-D.-Tankov (potential games) and Bertucci (non-potential games, under a strict monotoniciy condition).

Numerical algorithm

Linear programming fictious play algorithm The MFG problem is solved iteratively using the following algorithm

(i) Choose starting point $(\bar{m}^0, \bar{\mu}^0) \in \mathcal{R}$

(ii) For $n = 0, \ldots, N_{iter} - 1$

- Compute the best response

$$(\mu^{n+1}, m^{n+1}) = arg \max_{(\mu,m)\in\mathcal{R}} \Gamma[\bar{\mu}^n, \bar{m}^n](\mu, m).$$

- Update the measures:

$$(\bar{\mu}^{(n+1)}, \bar{m}^{(n+1)}) := \frac{n}{n+1}(\bar{\mu}^n, \bar{m}^n) + \frac{1}{n+1}(\mu^{(n+1)}, m^{(n+1)})$$
$$= \frac{1}{n+1}\sum_{l=1}^{n+1}(\mu^{(l)}, m^{(l)}).$$

To assess convergence, we monitor the "exploitability":

$$\mathcal{E}((\bar{\mu}^n, \bar{m}^n)) := \max_{(\mu,m)\in\mathcal{R}} \Gamma[\bar{\mu}^n, \bar{m}^n](\mu, m) - \Gamma[\bar{\mu}^n, \bar{m}^n](\bar{\mu}^n, \bar{m}^n).$$

Numerical algorithm

Convergence Assume that

- The conditions to have existence of an equilibrium are satisfied
- f satisfies the Lasry-Lions monotonicity condition with respect to m
- $\arg\max_{(\mu',m')\in R} \int_0^T \int_\Omega f(t,x,m)m'_t(dx)dt + \int_0^T \int_\Omega g(t,x,\mu)\mu'_t(dx)dt$ is unique up to dt-almost everywhere equivalence.

Theorem

Under the above assumption, the sequence $(\bar{\mu}^{(n)}, \bar{m}^{(n)})_{n\geq 1}$ converges to the unique MFG equilibrium in the topology $\tau_2^W \otimes \tau_2^M$.

Here, τ_p^W is the weak topology with respect to conttinous functions with p-growth and τ_p^M is the topology of convergence in measure on $M_p([0,T], \mathcal{P}_p^{sub}(\bar{\mathcal{O}}))$.

Outline

Entry/exit model for electricity markets

Application: entry/exit model for electricity markets

- We build a stylized equilibrium model of electricity market with conventional and renewable agents, interacting through the market price, allowing for entry and exit decisions (2 classes of agents).
- Conventional (e.g., gas) producers with fixed capacity and variable cost, aim to exit the market at the optimal time
- Renewable (e.g., wind) projects with variable capacity and zero marginal cost aim to enter the market at the optimal time
- The producers interact through the price resulting from a demand-supply equilibrium, which determines gains from production.
- Our goal: understand the effects of this interaction and of the market mechanisms on the long-term price levels and the renewable penetration.

Entry/exit model for electricity markets

Conventional producers

- Each **conventional producer** has marginal cost function $C_t^i : [0,1] \to \mathbb{R}$. $C_t^i(\xi)$ is the unit cost of increasing capacity if operating at ξ.
  We assume

$$C_t^i(\xi) = C_t^i + c(\xi),$$

where C_t^i is the baseline cost:

$$dC_t^i = k(\theta(t) - C_t^i)dt + \delta\sqrt{C_t^i}dW_t^i, \;\; C_0^i = c_i,$$

and $c : [0,1] \to \mathbb{R}$ is increasing smooth with $c(0) = 0$.

Entry-exit model for electricity markets

Conventional producers

- By maximizing its profit per unit, for a given price p, the producer offers fraction $F(p - C_t^i)$ of its capacity, where $F = c^{-1}$.
- Gain of the producer at price level p is $G(p - C_t^i)$, where

$$G(x) = \int_0^x F(z)dz, \ \ x \geq 0, \ \ G(x) = 0, \ \ x < 0.$$

Entry-exit model for electricity markets

Conventional producers

- Producer i aims to exit the market at the optimal time τ_i, where the optimization problem is

$$\max_{\tau} \mathbb{E}\left[\int_0^{T\wedge\tau} e^{-\rho t}(G(P_t - C_t^i) - \kappa_C)dt + K_C e^{-(\gamma_C+\rho)\tau\wedge T}\right],$$

where P_t is the electricity price, K_C is the cost of assets recovered upon exit, κ_C is the fixed running cost and γ_C is the depreciation rate.

Entry-exit model for electricity markets

Conventional producers

- The total conventional supply at price level p, including baseline conventional supply, is given by

$$\int_{\Omega} F(p-x)\omega_t^n(dx) + F_0(p) = \sum_{i=1}^{n} \frac{1}{n} F(p - C_t^i)\mathbf{1}_{\tau^i > t} + F_0(p),$$

with $\omega_t^n(dx)$ the distribution of costs of conventional producers who have not yet exited the market, i.e.

$$\omega_t^n(dx) = \frac{1}{n} \sum_{i=1}^{n} \delta_{C_t^i}(dx)\mathbf{1}_{\tau^i > t}.$$

Entry/exit model for electricity markets

Renewable producers

- Renewable producers aim to enter the market at the optimal time σ_i.
- To enter they pay the cost K_R after which the plant generates $S_t^i \in [0,1]$ units of electricity per unit of time at zero cost, where

$$dS_t^i = \bar{\kappa}(\bar{\theta} - S_t^i)dt + \bar{\delta}\sqrt{S_t^i(1-S_t^i)}d\overline{W}_t^i, \; S_0^i = s_i \in [0,1].$$

- The renewable producers always bid their full intermittent capacity and solve:

$$\max_{\sigma} \mathbb{E}\left[\int_{\sigma\wedge T}^{T} e^{-\rho t}(P_t S_t^i - \kappa_R)dt - K_R e^{-\rho\sigma_i\wedge T} + K_R e^{-\rho T - \gamma_R(T-\sigma\wedge T)}\right],$$

where K_R is the fixed cost, κ_R is the running cost and γ_R is the depreciation rate.

Entry-exit model for electricity markets

Renewable producers

- We denote by $\eta_t^n(dx)$ the distribution of output of renewable producers who have entered the market :

$$\eta_t^n(dx) = \frac{1}{n}\sum_{i=1}^{n} \delta_{S_t^i}(dx)\mathbf{1}_{\sigma_i \le t}.$$

- The total renewable supply at time t is given by $R_t^n = \int_0^1 x\eta_t^n(dx)$.

Entry/exit model for electricity markets

Price formation

- Agents are coupled through the market price, by matching exogenous demand process $\overline{D}_t$, to the aggregate supply function.

$$P_t := \inf\{P : (\overline{D}_t - R_t^n)^+ \leq \int_\Omega F(P - x)\omega_t^n(dx) + F_0(p)\} \wedge \overline{P},$$

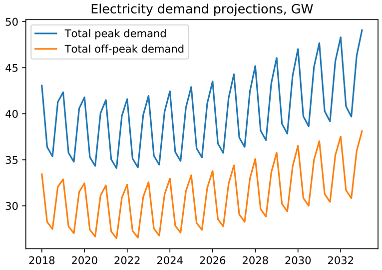
where $\overline{P}$ is the cap in the market.
When cap $\overline{P}$ is reached, demand is not entirely satisfied by producers.

Entry/exit model for electricity markets

Mathematical approach and results: MFG of optimal stopping, for which we use the linear programming formulation.

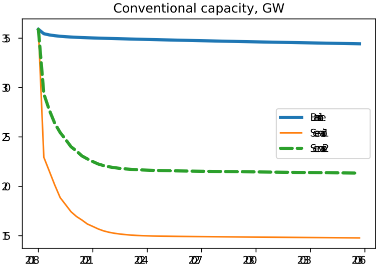
- The electricity market example requires extra mathematical work: two types of agents and interaction through the price (the price functional is highly irregular).
- We prove existence of Nash equilibrium and uniqueness of equilibrium price process.

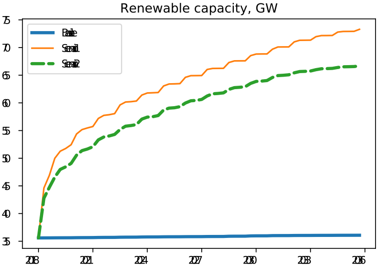
Numerical illustration: demand projections



We distinguish peak / off-peak price/demand for more realistic projections.

Numerical illustration: capacity





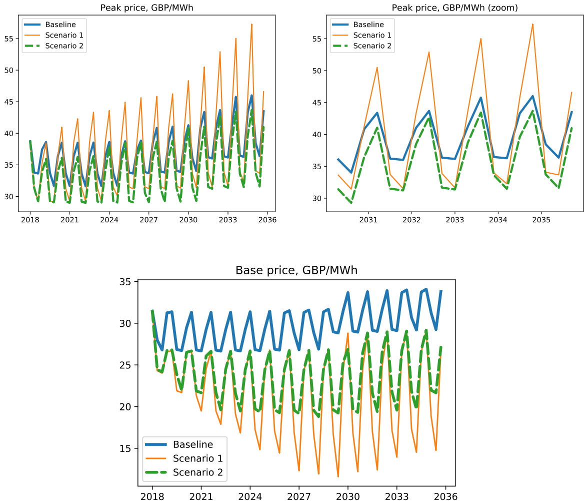
Conventional / renewable capacity evolution in three scenarios.

Baseline: costs estimated for UK market, no subsidy

Scenario 1: 30% renewable subsidy.

Scenario 2: renewable subsidy + a mechanism to keep conventional producers in the market.

Numerical illustration: price evolution



Entry-exit game for electricity markets

Extended model

- We consider a discrete-time version of the previous model and add a random carbon price. Study the impact on the pace of decarbonization of the electricity industry.
- Mathematical point of view: MFG of optimal stopping with common noise

Entry-exit game for electricity markets

Extended model

- Conventional producers $\mapsto$ Stochastic baseline cost $\mapsto$ decide when to exit the market
- Renewable producers $\mapsto$ Stochastic capacity factor $\mapsto$ decide when to enter the market.
- Carbon price impacts the cost of the conventional producers and the demand.
- Supplies from conventional and renewable producers $=$ Demand $\mapsto$ Electricity price.
- The optimization problems are coupled thorugh the electricity price $\mapsto$ Non cooperative game.
- We look for Nash equilibria.

Entry-exit game for electricity markets

Extended model

- The demand process is given by

$$D_t = d(t) + \beta(Z_t - Z_0),$$

where

  - $d(t)$ is a determistic function
  - $\beta \geq 0$: carbon price increases imply that carbon-intensive sectors of the industry are forced to electrify and contribute to electricity demand.

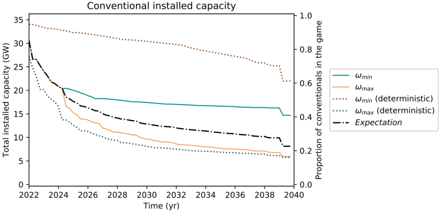
- The marginal unit cost of conventional producer i is given by

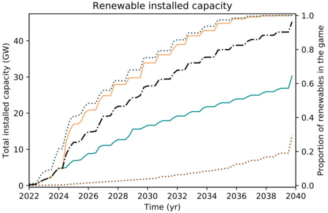
$$C_t^i(\xi) = C_t^i + \tilde{\beta} Z_t + c(\xi),$$

where

  - $\tilde{\beta} \geq 0$ represents the emission intensity

Entry-exit game for electricity markets





MFG for energy transition without common noise vs. with common noise

Additional developments

- We develop a discrete time optimal stopping MFG model which incorporates (possibly non-markovian) common noise and partial information
  - Existence of a *strong equilibrium*
  - Link between occupation measures and randomized stopping times and minimality property of the of the set of admissible measures
  - Construction of an approximate Nash equilibria for games with finite number of players
- The theory is applied to the previous model by incorporating common random shocks which affect the carbon price and the electricity demand. The shocks depend on a macroeconomic scenario which is not fully revealed to the agents

Outline

Control/stopping MFG

N-players game problem

- Consider N agents X^i, $i = 1, \dots, N$ with dynamics

$$dX_t^i = b(t, X_t^i, m_t^n, \alpha_t^i)dt + \sigma(t, X_t^i, m_t^n, \alpha_t^i)dW_t^i, \quad X_0^i \in \mathbb{R},$$

where W^i, $i = 1, \dots, N$ are independent and

$$m_t^n(dx, da) = \frac{1}{n} \sum_{k=1}^{N} \delta_{(X_t^k, \alpha_t^k)}(dx, da) \mathbf{1}_{t \leq \tau^k},$$

with (τ^k, α^k) a stopping time/regular control chosen by the player k.

Control/stopping MFG

N-players game problem

- Consider N agents X^i, $i = 1, \ldots, N$ with dynamics

$$dX^i_t = b(t, X^i_t, m^n_t, \alpha^i_t)dt + \sigma(t, X^i_t, m^n_t, \alpha^i_t)dW^i_t, \quad X^i_0 \in \mathbb{R},$$

where W^i, $i = 1, \ldots, N$ are independent and

$$m^n_t(dx, da) = \frac{1}{n} \sum_{k=1}^{N} \delta_{(X^k_t, \alpha^k_t)}(dx, da)\mathbf{1}_{t \le \tau^k},$$

with (τ^k, α^k) a stopping time/regular control chosen by the player k.

The control processes α take values in a compact set $A \subset \mathbb{R}$.

Control/stopping MFG

N-players game problem

- Each agent aims to solve the following *mixed control/optimal stopping problem*:

$$\sup_{\tau,\alpha} \mathbb{E}\left[\int_0^\tau f\left(t, X_t^k, m_t^n, \alpha_t^k\right) dt + g\left(\tau, X_\tau^k, \mu^n\right)\right]$$

where

$$\mu^n(dt, dx) = \frac{1}{n}\sum_{k=1}^n \delta_{(\tau^k, X_\tau^k)}(dt, dx).$$

Control/stopping MFG

MFG formulation

As $N \to \infty$, we expect (m^N, μ^N) converge to deterministic limits (m, μ).

- State process of the *representative agent*

$$dX_t^{\alpha,m} = b\left(t, X_t^{\alpha,m}, m_t, \alpha_t\right) dt + \sigma\left(t, X_t^{\alpha,m}, m_t, \alpha_t\right) dW_t,$$

- The mixed optimal stopping/control problem for the agent takes the form

$$\sup_{\tau,\alpha} \mathbb{E}\left[\int_0^\tau f\left(t, X^{\alpha,m}, m_t, \alpha_t\right) dt + g\left(\tau, X_\tau^{\alpha,m}, \mu\right)\right] \tag{5}$$

Control/stopping MFG

- Given the solution $(\tau^{\mu,m}, \alpha^{\mu,m})$ of the problem (5) for the agent facing a mean-field $(\mu, (m_t)_{t\in[0,T]})$, find $(\mu, (m_t)_{t\in[0,T]})$ such that

$$m_t(B) = \mathbb{P}\left[(X_t^{\alpha^{\mu,m},m}, \alpha_t^{\mu,m}) \in B, t \leq \tau^{\mu,m}\right], B \in \mathcal{B}(\mathbb{R}\times A), t \in [0,T], \tag{6}$$

and

$$\mu = \mathcal{L}\left(\tau^{\mu,m}, X_{\tau^{\mu,m}}^{\alpha^{\mu,m},m}\right). \tag{7}$$

Solution of the control/optimal stopping MFG: fixed point of (6) – (7).

MFG LP formulation

Fix a pair $(\bar{\mu}, \bar{m})$.

- Define $\mathcal{R}[\bar{m}]$ as the set of pairs (μ, m), such that for all $u \in C_b^{1,2}([0, T] \times \mathbb{R})$,

$$\int_{[0,T]\times\mathbb{R}} u(t,x)\mu(dt,dx) = \int_{\mathbb{R}} u(0,x)m_0^*(dx) + \int_0^T \int_{\mathbb{R}\times A} (\partial_t u + \mathcal{L}u)(t,x,\bar{m}_t,a)m_t(dx,da)dt,$$

$$\mathcal{L}u(t,x,\bar{m}_t,a) = b(t,x,\bar{m}_t,a)\partial_x u(t,x) + \frac{\sigma^2}{2}(t,x,\bar{m}_t,a)\partial_{xx}u(t,x).$$

- Let $\Gamma[\bar{\mu}, \bar{m}] : \mathcal{P}([0, T] \times \mathbb{R}) \times V \to \mathbb{R}$ be defined by

$$\Gamma[\bar{\mu},\bar{m}](\mu,m) = \int_0^T \int_{\mathbb{R}\times A} f(t,x,\bar{m}_t,a)m_t(dx,da)dt + \int_{[0,T]\times\mathbb{R}} g(t,x,\bar{\mu})\mu(dt,dx).$$

MFG LP formulation

- We say that $(\mu^\star, m^\star)$ is an *LP MFG Nash equilibrium* if $(\mu^\star, m^\star) \in \mathcal{R}[m^\star]$ and for all $(\mu, m) \in \mathcal{R}[m^\star]$,

$$\Gamma[\mu^\star, m^\star](\mu, m) \leq \Gamma[\mu^\star, m^\star](\mu^\star, m^\star).$$

The real number $\Gamma[\mu^\star, m^\star](\mu^\star, m^\star)$ is called *Nash value*.

Existence of LP MFG equilibria

- One can construct a space $\mathcal{R}_0$ with good mathematical properties such that all the sets $\mathcal{R}[\bar{m}]$ are included in it.
- We introduce the set valued map $\mathcal{R}^\star : \mathcal{R}_0 \to 2^{\mathcal{R}_0}$ as

$$\mathcal{R}^\star(\bar{\mu}, \bar{m}) = \mathcal{R}[\bar{m}].$$

- Define $\Theta : \mathcal{R}_0 \to 2^{\mathcal{R}_0}$ as

$$\Theta(\bar{\mu}, \bar{m}) = \underset{(\mu,m)\in\mathcal{R}^\star(\bar{\mu},\bar{m})}{\arg\max} \Gamma[\bar{\mu}, \bar{m}](\mu, m).$$

$\Rightarrow$ the set of Nash equilibria coincides with the set of fixed points of the set-valued mapping Θ

Existence of LP MFG equilibria

Theorem

The set of LP MFG Nash equilibria is compact and nonempty.

(1) Prove that the set-valued mapping $\mathcal{R}^\star$ is continuous in the sense of set-valued mappings (lower and upper hemicontinuous)

(2) By Berge Maximum Theorem, get that the set valued mapping Θ is upper hemicontinuous and has nonempty compact values

(3) Apply Kakutani-Fan-Glicksberg Theorem to get the existence of an equilibrium.

Existence of LP MFG equilibria

(1) By the disintegration theorem, for each $(m_t)_{t\in[0,T]} \in V$, there exists a mapping $\nu_{t,x} : [0,T] \times \mathbb{R} \to \mathcal{P}(A)$ such that for each $B \in \mathcal{B}(A)$, the function $(t,x) \mapsto \nu_{t,x}(B)$ is $\mathcal{B}([0,T] \times \mathbb{R})$-measurable, and

$$m_t(dx,da)dt = \nu_{t,x}(da)m_t(dx,A)dt,$$

where $m_t(dx,A) := \int_A m_t(dx,da)$. $(\nu_{t,x}(\cdot))$ is called *Markovian relaxed control*.

(2) MFG LP equilibria taking the form $m_t(dx,da) = \delta_{\alpha(t,x)}(da)m_t(dx,A)$ for some measurable function $\alpha : [0,T] \times \mathbb{R} \to A$ are called *strict control MFG equilibria*.

Under the convexity assumption of the set $K[m](t,x) := \{(b(t,x,m_t,a), \sigma^2(t,x,m_t,a), z) : a \in A, z \leq f(t,x,m_t,a)\}$, we get the existence of a **strict control LP MFG equilibrium**.

Control/stopping MFG

Results

- **Develop** the **linear programming approach** for the general case of *mixed stochastic control and optimal stopping* and coefficients (b, σ) which depend on the measure.
- Develop fictious linear programming algorithm for **MFG with pure control and absorption**
- Establish the **link** between the existence of an LP MFG Nash equilibrium and the existence of MFG Nash equilibrium via the **controlled/stopped martingale problem** (used before in the case when there is only control).
- Establish the **link** with the **PDE approach**.

Thank you for your attention!