Exact simulation of the Hull and White stochastic volatility model

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Introduction

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- It is useful since when using time discretization it is not possible to determine *a priori* the number of time steps needed to reduce the discretization bias to an acceptable level;
- Since the bias is unknown, the standard error may be a poor estimate of the actual error, and valid confidence intervals are not available.

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- Exact simulation of stochastic volatility models relies on nested-conditional factorization: suppose to have two dependent random variables X and Y and one is interested to obtain a joint sample (X, Y), one simulates Y first and then X conditionally on Y. Various stochastic volatility models can be simulated exactly by choosing appropriately X and Y;

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- The main advantage of exact simulation is that the convergence rate for exact simulation schemes is $O(s^{-1/2})$, where s is the total computational budget, whereas the convergence rate for discretization methods is slower. For example, for the Euler scheme it is $O(s^{-1/3})$.

Literature Review (I)

Consider a general stochastic volatility model

$$dS_t = rS_t dt + \sigma_S(S_t, V_t, t)(\rho dB_t + \sqrt{1-\rho^2} dW_t)$$
$$dV_t = \mu_V(V_t, t)dt + \sigma_V(V_t, t)dB_t$$

where B_t and W_t are independent standard Brownian motions, r is the risk-less rate and $\rho \in [-1, 1]$.

Model	$\sigma_S(S_t, V_t, t)$	$\mu_V(V_t, t)$	$\sigma_V(V_t, t)$	Y	X
Heston	$S_t\sqrt{V_t}$	$k(\theta - V_t)$	$\sigma\sqrt{V_t}$	V_T	$\int_0^T V_s ds$
3/2	$S_t \frac{b}{\sqrt{V_t}}$	$k(\theta - V_t)$	$\sigma\sqrt{V_t}$	V_T	$\int_0^T \frac{1}{V_s} ds$
SABR	$S_t^\beta \sqrt{V_t}$	$\sigma^2 V_t$	$2\sigma V_t$	V_T	$\int_0^T V_s ds$
4/2	$S_t \left(a\sqrt{V_t} + \frac{b}{\sqrt{V_t}}\right)$	$k(\theta - V_t)$	$\sigma\sqrt{V_t}$	V_T	$\log S_T$
SV-OU	$S_t\sqrt{V_t}$	$2k\left(\frac{\sigma^2}{2k} + \theta\sqrt{V_t} - V_t\right)$	$2\sigma\sqrt{V_t}$	$\left(V_T, \int_0^T \sqrt{V_s} ds\right)$	$\int_0^T V_s ds$
HW-SV	$S_t\sqrt{V_t}$	ηV_t	σV_t	V_T	$\int_0^T \sqrt{V_s} ds$

Notes. Relevant literature: Broadie and Kaya (2006, OR), Baldeaux (2012, IJTAF), Cai et al. (2017, OR), Grasselli (2017, MaFi), Li and Wu (2019, EJOR).

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- This precludes the possibility of pricing European options through standard Fourier inversion techniques (e.g. FFT and COS methods);
- Pricing can be done by simulating discrete approximations, making the procedure time-consuming and introducing noticeable bias;
- Ackerer and Filipovic (2020, MaFi) find that the erratic behavior of the moments renders polynomial option pricing techniques more delicate to apply with respect to more standard models;
- Zeng et al. (2023, MaFi) provide a general framework for the exact simulation of stochastic volatility models, but conclude that the HW-SV model can not be simulated exactly through their method, and that only approximations are possible.

Main Contributions

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- We propose a conditional COS formula for pricing European options, which is useful in reducing the variance of the Monte Carlo estimator of the option price and the computing time;
- Through a second variant of our simulation scheme, we derive unbiased estimates for the Greeks of European call options;

Exact Simulation Scheme

HW-SV: conditional distribution of the log-price

Given a final date $T > 0$, the conditional log-asset price is given as follows:

$$\left(\ln S_T \,\middle|\, V_T, \int_0^T V_s ds, \int_0^T \sqrt{V_s} ds, V_0, S_0\right) \sim \mathcal{N}(m, s^2) \qquad (1)$$

where

$$m = \ln(S_0) + rT - \frac{\int_0^T V_s ds}{2} + \frac{\rho}{v}\left(\sqrt{V_T} - \sqrt{V_0} - \frac{(\eta - v^2)}{2}\int_0^T \sqrt{V_s} ds\right)$$

$$V_T = V_0 \exp\left(\left(\eta - \frac{\sigma^2}{2}\right)T + \sigma B_T\right), \quad s^2 = (1-\rho^2)\int_0^T V_s ds, \quad v = \frac{\sigma}{2}.$$

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Our benchmark is given as follows. First, we divide the time interval $[0, T]$ into n equispaced points of length T/n. Then, we simulate trajectories of the variance exactly. Finally, we approximate the integrated variance and integrated volatilities using the trapezoidal rule and the asset price from (1).

Main idea (I)

The proposed exact simulation algorithm relies on a conditional nested factorization approach and can be summarized in three main steps:

- Step 1: simulate V_T
- Step 2: simulate $\left(\frac{1}{\int_0^T \sqrt{V_s} ds} \middle| V_T\right)$ and recover $\left(\int_0^T \sqrt{V_s} ds \middle| V_T\right)$ by taking the reciprocal
- Step 3: simulate $\left(X_T \middle| \int_0^T \sqrt{V_s} ds, V_T\right)$, where $X_T := \log(S_T/S_0)$

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The difference with respect to the approach in Zeng et al. (2023, MaFi) is that they bypass Step 2 and attempt sampling directly $\left(X_T \middle| V_T\right)$ which is not possible since the relevant Laplace transform is unknown, while the Laplace transform of $\left(X_T \middle| \int_0^T \sqrt{V_s} ds, V_T\right)$ is known (as we illustrate later).

Main idea (II)

- Step 1 is straightforward since V is lognormally distributed
- Step 2 is implemented by numerically inverting the Laplace transform of $\left(\frac{1}{\int_0^T \sqrt{V_s}ds}\middle| V_T\right)$ to the cumulative distribution function and then sampling using inverse transform method
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$\mathcal{L}_1(u)$: Laplace transform of $\left(\frac{1}{\int_0^T \sqrt{V_s} ds} \middle| V_T\right)$

Proposition
Given $u > 0$, we have

$$\mathcal{L}_1(u) := \mathrm{E}\left[\exp\left(-\frac{u}{\int_0^T \sqrt{V_s} ds}\right) \middle| V_T\right]$$

$$= \exp\left(-\frac{\varphi\left(\frac{1}{4}\log\frac{V_T}{V_0}, \frac{u\sigma^2}{16\sqrt{V_0}}\right)^2 - \left(\frac{1}{4}\log\frac{V_T}{V_0}\right)^2}{T\frac{\sigma^2}{8}}\right) \tag{2}$$

where $\varphi(x, \lambda) = \mathrm{arcosh}(\lambda e^{-x} + \cosh(x))$

Proposition

Given $u > 0$, we have

$$\begin{aligned}\mathcal{L}_2(u) &:= \mathrm{E}\left[\exp\left(-u\int_0^T V_s ds\right) \Big| \int_0^T \sqrt{V_s} ds, V_T\right] \\ &= \frac{\theta(\phi(v^\star, x^\star, \sqrt{\lambda^\star}), t^\star/4)}{\psi_{t^\star}^{(\mu^\star)}(v^\star, x^\star)} e^{\mu^\star x^\star - (\mu^\star)^2 t^\star/2} \times \\ &\times \frac{\sqrt{\lambda^\star}}{4\sinh(\sqrt{\lambda^\star} v^\star/2)} \exp\left(-\sqrt{\lambda^\star}(1+e^{x^\star})\coth(\sqrt{\lambda^\star} v^\star/2)\right)\end{aligned} \tag{3}$$

where

$$\phi(v, x, \lambda) = \frac{2\lambda\exp(x/2)}{\sinh(\lambda v/2)}, \quad \psi_t^{(\mu)}(v, x) = \frac{1}{2v} e^{\mu x - \mu^2 t/2} \exp\left(-\frac{2(1+x)}{v}\right)\theta(4e^{x/2}/v, t/4)$$

$$\theta(r, t) = \frac{r}{\sqrt{2\pi^3 t}} \int_0^\infty e^{-\frac{\xi^2}{2t}} e^{-r\cosh(\xi)} \sinh(\xi) \sin\left(\frac{\pi\xi}{t}\right) d\xi$$

and $\lambda^\star = \frac{8uV_0}{\sigma^2}$, $t^\star = T\frac{\sigma^2}{4}$, $x^\star = \frac{1}{2}\log\frac{V_T}{V_0}$ *and* $v^\star = \frac{\int_0^T \sqrt{V_s} ds}{\sqrt{V_0}\frac{4}{\sigma^2}}$, $\mu^\star = \frac{2\eta}{\sigma^2} - 1$

$\mathcal{L}_3(u)$: Laplace transform of $\left(X_T | V_T, \int_0^T \sqrt{V_s} ds\right)$

Proposition
Given $u > 0$, we have

$$\mathcal{L}_3(u) := \mathrm{E}\left[e^{-uX_T}\Big| V_T, \int_0^T \sqrt{V_s} ds\right]$$
$$= \exp\left[-u\left(rT + \frac{\rho}{v}\left(\sqrt{V_T} - \sqrt{V_0} - \frac{1}{2}(\eta - v^2)\int_0^T \sqrt{V_s} ds\right)\right)\right]$$
$$\times \mathcal{L}_2\left(-\left(\frac{1}{2} + \frac{1}{2}u(1-\rho^2)\right)\right) \quad (4)$$

Random Sampling using COS

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- According to the COS method, the pdf of a generic r.v. is given as

$$f_j(y) = \sum_{k=1}^{\infty} F_k \cos\left(k\pi \frac{y-a}{b-a}\right) + \frac{1}{b-a} \approx \sum_{k=1}^{N-1} F_k \cos\left(k\pi \frac{y-a}{b-a}\right) + \frac{1}{b-a}$$

where $F_k = \frac{2}{b-a} \mathrm{Re}\left(\mathcal{F}_j\left(\frac{k\pi}{b-a}\right) \cdot \exp\left(-i\frac{ka\pi}{b-a}\right)\right)$ and $[a, b] \in \mathcal{D}$ is chosen such that:

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$$c_j(y) \approx \int_a^y f_j(h) dh = \frac{y-a}{b-a} \sum_{k=1}^{N-1} F_k \frac{(b-a)\sin\left(\frac{\pi k(a-y)}{a-b}\right)}{\pi k} \qquad (6)$$

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- We sample $U \sim \mathcal{U}(0, 1)$ and find y such that $c_j(y) = U$ using root finding algorithms.

Control of the error: Truncation range computation (I)

- If the interval $[a, b]$ is too wide, then one needs a larger N to get sufficient accuracy; if the interval is too tight, the approximation in (5) may be inaccurate

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- Since X_T is defined on $\mathbb{R}$, for $\left(X_T | \int_0^T \sqrt{V_s} ds, V_T\right)$ we can choose a and b according to Fang and Osterlee (2008):

$$a = \tilde{c}_1 - 12\sqrt{\tilde{c}_2}, \quad b = \tilde{c}_1 + 12\sqrt{\tilde{c}_2} \tag{7}$$

where $\tilde{c}_j$ denotes the j-th cumulant of the risk–neutral distribution of log–returns

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- Since $\left(\int_0^T \sqrt{V_s} ds | V_T\right) \in \mathbb{R}^+$, the choice is based on the first two integer conditional moments of $\left(\frac{1}{\int_0^T \sqrt{V_s} ds} | V_T\right)$

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- In both cases, we can use the results in Kyriakou et al. (2023, OR) to compute the moments of some unknown distribution given the knowledge of the Laplace transform

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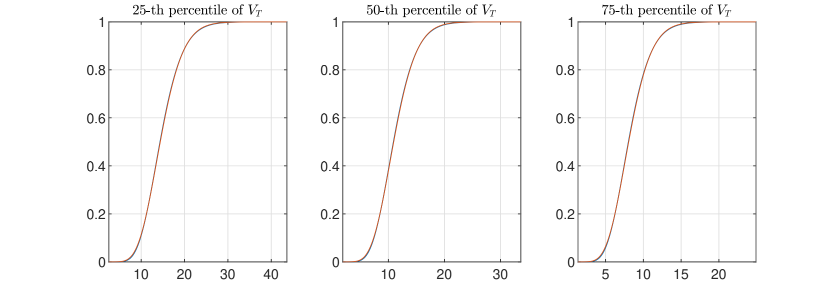
- Hence, we choose a and b in (5) in such a way that the probability of $\mathcal{G} \leq a$ is 10^{-12} and probability that $\mathcal{G} \leq b$ is $1 - 10^{-12}$
- When the truncation range is computed accurately, the overall error is controlled by the parameter N

Control of the error: Truncation range computation (III)

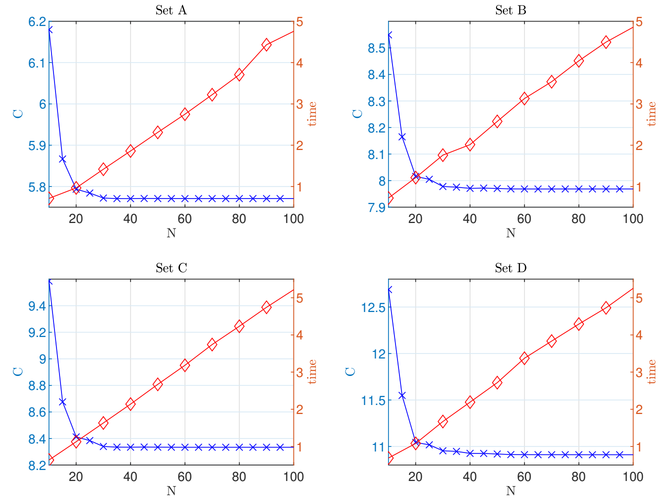
Figure 1: Truncation range and cumulative distribution function of $\left(\frac{1}{\int_0^T \sqrt{V_s} ds} \middle| V_T\right)$.



Notes. Cumulative distribution functions of the true distribution (blue line) and the moment–matched Gamma distribution (red line) for three different values of V_T. x–axis is truncated at a and b.

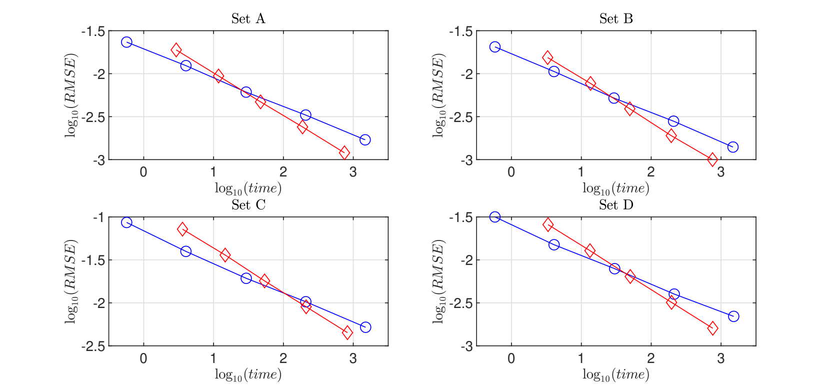
Control of the error: Convergence to the true option price

Figure 2: Option price convergence (blue) and runtime (red).



Numerical results: Exact simulation vs benchmark

Figure 3: Speed-accuracy comparisons of our exact simulation scheme and competent benchmark for different parameter sets: the case of European plain vanilla option.



Notes. Our exact simulation algorithm: plots with red diamond markers; benchmark: plots with blue circle markers. All computing times are expressed in seconds.

Option pricing and Greeks evaluation

Option Pricing using Conditional COS

Instead of simulating $\left(X_T | \int_0^T \sqrt{V_s} ds, V_T\right)$, we can compute directly the conditional option price, i.e. the option price conditional on the random realization of $\left(\int_0^T \sqrt{V_s} ds, V_T\right)$:

$$
\begin{aligned}
C &\approx e^{-rT}\frac{1}{b-a}\int_{\log\left(\frac{K}{S_0}\right)}^{b}(S_0e^y - K)dy + e^{-rT}\sum_{k=1}^{N-1}F_k\int_{\log\left(\frac{K}{S_0}\right)}^{b}(S_0e^y - K)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \\
&= \frac{e^{-rT}}{b-a}\left(-(b+1)K + e^bS_0 + K\log\left(\frac{K}{S_0}\right)\right) + \sum_{k=1}^{N-1}F_k\frac{e^{-rT}\left(-(b+1)K + e^bS_0 + K\log\left(\frac{K}{S_0}\right)\right)}{b-a} \\
&+ (a-b)e^{-rT}\left(\frac{-e^bS_0((b-a)\cos(\pi k) + \pi k\sin(\pi k)) + K(b-a)\cos(\zeta) + \pi kK\sin(\zeta)}{(a-b)^2 + \pi^2k^2} + \right. \\
&+ \left.\frac{K(\sin(\pi k) - \sin(\zeta))}{\pi k}\right), \qquad (8)
\end{aligned}
$$

where $\zeta = \frac{\pi k\left(a - \log\left(\frac{K}{S_0}\right)\right)}{a-b}$.

Greeks evaluation using Conditional COS

In the same way, we can also compute Greeks. For example, the "Delta" is computed according to

$$\Delta = \frac{\partial C}{\partial S_0} = \frac{e^{-rT}\left(e^b - \frac{K}{S_0}\right)}{b-a} + \sum_{k=1}^{N-1} F_k(a-b)e^{-rT} \times$$
$$\times \left(\frac{\frac{\pi^2 k^2 K\cos(\zeta)}{S_0(a-b)} - \frac{\pi k K(b-a)\sin(\zeta)}{S_0(a-b)} - e^b((b-a)\cos(\pi k) + \pi k \sin(\pi k))}{(a-b)^2 + \pi^2 k^2} - \frac{K\cos(\zeta)}{S_0(a-b)} \right). \quad (9)$$

where $\zeta = \frac{\pi k\left(a - \log\left(\frac{K}{S_0}\right)\right)}{a-b}$.

Computational advantages of Conditional COS

- This approach (which we label *conditional COS*) presents an important advantage with respect to standard exact simulation algorithm: since we don't sample X_T, we remove the additional Monte Carlo variance due to Step 3.

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- It can be used as a variance reduction technique when the purpose is to price European options, allowing to obtain unbiased estimates with tight confidence interval at a small computational cost. In the course of our numerical studies, we find that the variance of the Monte Carlo estimator is reduced by roughly 93-98%.

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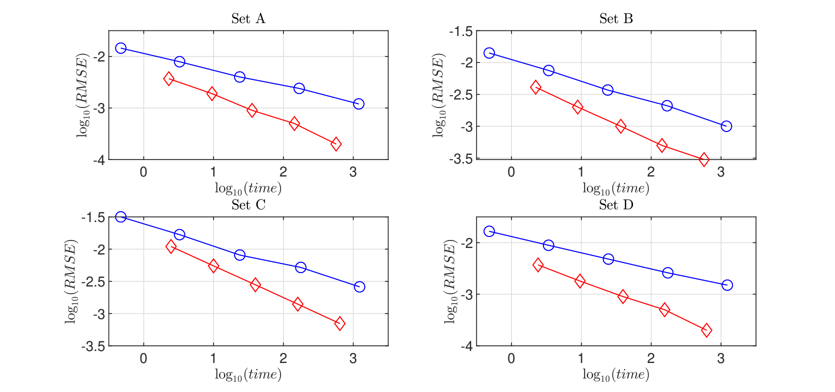
- Further, we observe a reduction of the running time because we can directly price the option without simulating $\left(X_T | \int_0^T \sqrt{V_s} ds, V_T\right)$, avoiding root finding algorithms. For example, for 1024×10^4 simulations the reduction is approximately 24%.

Numerical results: conditional COS vs benchmark

Figure 4: Speed-accuracy comparisons of conditional COS method and competent benchmark for different parameter sets: the case of European plain vanilla option.



Notes. Conditional COS: plots with red diamond markers; benchmarks: plots with blue circle markers. All computing times are expressed in seconds.

Conclusions

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- We propose some variants to the exact simulation scheme that allow to reduce the computational time and the variance of the Monte Carlo estimator when computing European call option prices and Greeks;
- The proposed approaches present a faster convergence rate of the Root Mean Squared Error (RMSE) with respect to the benchmark.

Thank you!