Optimal Execution with Relative Entropy

a Schrödinger Bridge Approach

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Market Impact of Order Execution

- Market impact: Initiating a buy order will drive the price up and initiating a sell order will drive the price down
- Brokers often face the problem of liquidating a large position within a relatively short period of time on clients' behalf

Splitting order

- Splitting a metaorder into a series of child orders
  - To reduce market impact: trading slowly
  - But a longer execution period incurs an extra exposure to the risk of price movement: trading quickly
- Execution strategy: A method to split the order: a trading curve from q_0 to q_T

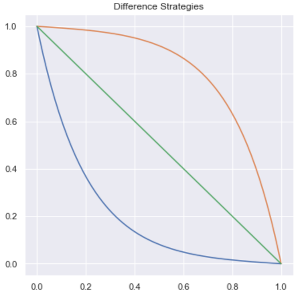


Figure: Execution Strategies

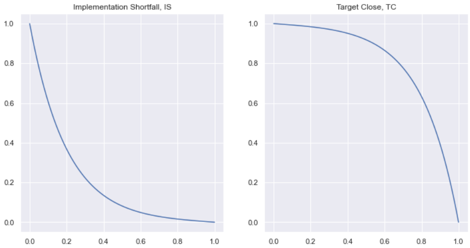
Reservation Strategy

- In short, the broker has a "prior preference" before trading, described by a reservation strategy process
- Intuitively, if there is no limitation, the broker will follow the reservation strategy

Reservation Strategy

- Many strategies also consider trade-cost risk with different benchmark strategies (Cheng, Guo, and Wang 2024)
  - IS orders: benchmarked to a block trading at the beginning
  - TC orders: benchmarked to a block trading at the end of the trading day
  - VWAP/TWAP orders: following the market flow

$$R_t^{\mathrm{IS}} = \begin{cases} q_0, & t = 0, \\ 0, & 0 < t \le 1. \end{cases} \quad , \quad R_t^{\mathrm{TC}} = \begin{cases} q_0, & 0 \le t < 1, \\ 0, & t = 1. \end{cases}$$



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Frei and Westray 2015 propose a Gamma bridge as the relative volume process (reservation strategy), where the associated stochastic control problem is given by

$$\min_{(v_t)} \mathbb{E}\left[F q_1^2 + \int_0^T \left(\underbrace{\lambda\sigma^2 (q_t - \gamma_t)^2}_{\text{distance}} + \eta v_t^2 \right) \mathrm{d}t \right], \tag{1}$$

$$\text{s.t. } q_0 = 1,\ \mathrm{d}q_t = v_t \mathrm{d}t, \tag{2}$$

where γ_t is a Gamma bridge, q_t is the inventory process and v_t is the trading velocity. In this model, the optimal trading curve q_t^* is also stochastic.

Model Setup

- Consider the multi-asset optimal execution problem: from q_0 to q_T during $[0, T]$
  - $q_t \in \mathbb{R}^d$: the inventory process
  - $R_t \in \mathbb{R}^d$: the reservation strategy process
- We use dynamic relative entropy (or KL divergence) to describe the "distance" between q and R:
  - Suppose the measures of q_t and R_t are respectively $\mathbb{P}_t$ and $\mathbb{Q}_t$
  - KL divergence

$$D_{\mathrm{KL}}(\mathbb{P}_t||\mathbb{Q}_t) := \int_{\mathbb{R}^d \times \mathbb{R}^d} \log \frac{\mathrm{d}\mu^{\mathbb{P}}}{\mathrm{d}\mu^{\mathbb{Q}}} \mu^{\mathbb{P}}(\mathrm{d}x\mathrm{d}y)$$

- More precisely, we have the Schrödinger Bridge Problem

Schrödinger Bridge Problem

- Let $\xi = \mathcal{N}(\mu_0, \Sigma_0)$ and $\xi' = \mathcal{N}(\mu_1, \Sigma_1)$ be given Gaussian measures
- The Schrödinger Bridge Problem refers to the constrained dynamic entropy-regularized problem over all stochastic processes $\mathbb{P}_t$ (Chen, Georgiou, and Pavon 2021, Bunne et al. 2023)

$$\min_{\mathbb{P}_0=\xi,\mathbb{P}_1=\xi'} D_{\mathsf{KL}}(\mathbb{P}_t||\mathbb{Q}_t)$$

- ξ and ξ' are the (empirical) marginal distributions of a complicated continuous-time dynamics observed at the starting and end times
- $\mathbb{Q}_t$ is a "prior process" representing our belief of the dynamics before observing any data.

Schrödinger Bridge Problem

- Consider the Schrödinger Bridge Problem

$$\min_{\mathbb{P}_0=\xi,\mathbb{P}_1=\xi'} D_{\mathrm{KL}}(\mathbb{P}_t||\mathbb{Q}_t). \tag{3}$$

  We take $\mathbb{Q}_t$ as the measure of a linear SDE

$$\mathrm{d}R_t = A_t R_t \mathrm{d}t + B_t \mathrm{d}W_t.$$

- Denote $\tau_t := \exp\left(\int_0^t A_s \mathrm{d}s\right)$ and suppose the mean function of R_t is

$$\mathbb{E}[R_t|R_0] = \tau_t R_0 =: \eta(t)$$

  and the covariance function is given by $(t' \geq t)$

$$\mathrm{Cov}[R_t, R_{t'}|Y_0] = \left(\tau_t \tau_{t'} \int_0^t \tau_s^2 B_s^2 \mathrm{d}s\right) I =: \kappa(t, t')I.$$

Parameters

Set

$$D_\sigma = \left(4\Sigma_0^{\frac{1}{2}}\Sigma_1\Sigma_0^{\frac{1}{2}} + \sigma^4 I\right)^{\frac{1}{2}},\ C_\sigma = \frac{1}{2}\left(\Sigma_0^{\frac{1}{2}} D_\sigma \Sigma_0^{-\frac{1}{2}} - \sigma^2 I\right),$$

$$r_t = \frac{\kappa(t,1)}{\kappa(1,1)},\ \bar{r}_t = \tau_t - r_t\tau_1,\ \sigma_* = \sqrt{\tau_1^{-1}(1,1)},$$

$$\rho_t = \frac{\int_0^t \tau_s^2 B_s^2 \mathrm{d}s}{\int_0^1 \tau_s^2 B_s^2 \mathrm{d}s},$$

$$P_t = \dot{r}_t(r_t\Sigma_1 + \bar{r}_t C_{\sigma_*}),\ Q_t = -\dot{\bar{r}}_t(\bar{r}_t\Sigma_0 + r_t C_{\sigma_*}),$$

$$S_t = P_t - Q_t^\top + \left(A_t\kappa(t,t)(1-\rho_t) - g_t^2\rho_t\right)$$

Optimal Solution

By using the results in Bunne et al. 2023, we may solve the Schrödinger Bridge Problem:

Proposition

1. *The solution to the Schrödinger Bridge Problem is a Markov Gaussian process whose marginal variable* $q_t \sim \mathcal{N}(\mu_t, \Sigma_t)$, *where*

$$\mu_t = \bar{r}_t\mu_0 + r_t\mu_1,$$

$$\Sigma_t = \bar{r}_t^2\Sigma_0 + r_t^2\Sigma_1 + r_t\bar{r}_t(C_{\sigma_*} + C_{\sigma_*}^\top) + \kappa(t,t)(1-\rho_t)I.$$

2. q_t *admits a closed-form representation as the SDE*

$$dq_t = \left(S_t^\top \Sigma_t^{-1}(q-\mu_t) + \dot{\mu_t}\right) dt + B_t dW_t.$$

Equivalent Stochastic Control Problem

Chen, Georgiou, and Pavon 2015 shows that the Gaussian Schrödinger bridge problem (3) is equivalent to the following stochastic control problem:

$$\min_{(u_t)} \mathbb{E}\left[\int_0^1 ||u_t||^2 \mathrm{d}t\right] \tag{4}$$

$$\text{s.t. } \mathrm{d}x_t = (A_t x_t + u_t)\,\mathrm{d}t + B_t \mathrm{d}W_t, \tag{5}$$

$$x_0 \sim \mathcal{N}(\mu_0, \Sigma_0), \quad x_1 \sim \mathcal{N}(\mu_1, \Sigma_1). \tag{6}$$

Note that the dynamic of R_t is given by

$$\mathrm{d}R_t = A_t R_t \mathrm{d}t + B_t \mathrm{d}W_t,$$

so R_t is the "uncontrolled" process.

Almgren-Chriss Framework with Execution Risk

- While executing a sequence of pre-scheduled orders, the orders in the sequence may not be fully executed
- Carmona and Leal 2023 show that there exists a Brownian component in the inventory processes of individual traders by statistical tests
- Cheng, Di Giacinto, and Wang 2017 assume that

$$\mathrm{d}q_t = v_t \mathrm{d}t + \underbrace{m_0 \mathrm{d}Z_t}_{\text{execution risk}},$$

instead of $v_t = \frac{\mathrm{d}q_t}{\mathrm{d}t}$

Almgren-Chriss Framework with Execution Risk

Example

The risk-neutral stochastic control problem in Cheng, Di Giacinto, and Wang 2017 is given by

$$\min_{(v_t)} \mathbb{E}\left[\int_0^1 v_t^2 \mathrm{d}t\right], \tag{7}$$

$$\text{s.t.}\ \ \mathrm{d}x_t = v_t \mathrm{d}t + m_0 \mathrm{d}W_t, \tag{8}$$

$$x_0 = X,\ \ x_1 = 0. \tag{9}$$

The stochastic control problem is equivalent to the Schrödinger Bridge Problem, with reservation strategy

$$\mathrm{d}R_t = m_0 \mathrm{d}W_t.$$

State Penalty

To introduce the state penalty, we generalize a proposition in Chen and Georgiou 2015:

Proposition

Denote x_t^* by the optimal solution to the stochastic control problem with $F \to \infty$:

$$\min_{(u_t)} \mathbb{E}\left[x_1' F x_1 + \int_0^1 \left(x_t' Q_t x_t + ||u_t||^2\right) dt\right] \tag{10}$$

$$s.t.\ \ dx_t = (A_t x_t + u(t))\, dt + B_t dW_t,\ \ 0 < t < 1,\ \ x_0 = X. \tag{11}$$

Suppose x_t is the solution to the SDE

$$dx_t = A_0(t) x_t dt + B_t dW_t,\ \ 0 < t < 1,\ \ x_0 = X,$$

where A_0 solves the ODE $\dot{A}_0 + A_0^\top A_0 - \dot{A} - A^\top A - Q = 0.$

State Penalty

Proposition

Furthermore, suppose x_t^{**} is the optimal solution to the stochastic control problem with $F \to \infty$

$$\min_{(u_t)} \mathbb{E}\left[x_1' F x_1 + \int_0^1 ||u_t||^2 dt\right] \tag{12}$$

$$s.t.\ \ dx_t = (A_0(t)x_t + u_t)\, dt + B_t dW_t,\ 0 < t < 1,\ x_0 = X. \tag{13}$$

Then x_t^*, x_t^{**} and $x_t|\{x_1 = 0\}$ have the same distribution. Moreover, they solve the Schrödinger Bridge Problem, with reservation strategy given by

$$dR_t = A_0(t)R_t dt + B_t dW_t,\ 0 < t < 1,\ R_0 = x_0.$$

Almgren-Chriss Framework with Execution Risk

Example

The mean-quadratic variation stochastic control problem in Cheng, Di Giacinto, and Wang 2017 is given by

$$\min_{(v_t)} \mathbb{E}\left[\int_0^1 \left(v_t^2 + \frac{\lambda\sigma^2}{\eta} q_t^2\right) \mathrm{d}t\right], \tag{14}$$

$$\text{s.t. } \mathrm{d}x_t = v_t \mathrm{d}t + m_0 \mathrm{d}W_t, \tag{15}$$

$$x_0 = X,\ x_1 = 0. \tag{16}$$

The stochastic control problem is equivalent to the Schrödinger Bridge Problem, with an OU process as reservation strategy

$$\mathrm{d}R_t = -\sqrt{\frac{\lambda\sigma^2}{\eta}} R_t \mathrm{d}t + m_0 \mathrm{d}W_t.$$

Conclusion

- We formulate the broker's problem as a dynamic relative entropy optimization problem.
- We derive the explicit form of the Schrödinger Bridge Problem.
- We consider the case where the density constraint is replaced by a fixed-point constraint, and the associated state process reduces to a pinned process, which is a generalization of the Brownian bridge to linear systems.
- We show that our model can cover Almgren-Chriss framework with execution risk by setting different reservation strategies.

THANK YOU

FOR YOUR ATTENTION !