A Comparison Study of ADI and ADE Methods of the Black-Scholes equation on option pricing models

Neda Bagheri Renani

Doctoral advisor: Daniel Ševčovič.

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Interest rate and Statistic Black-Scholes equation

If we assume that

$$dr = \kappa(\theta - r)dt + \sigma_r dw_r$$

and

$$dS = \mu S dt + \sigma S dw_s$$

Where

$$Cor(dw_s, dw_r) = \rho.dt$$

and

$$\frac{\partial V}{\partial t} = \frac{\sigma_s^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV$$

S is geometric Brownian motion

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{\sigma_s^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 V}{\partial r^2} + \rho\sigma r S \frac{\partial^2 V}{\partial S \partial r} + (\kappa(\theta - r) - \lambda\sigma_r)\frac{\partial V}{\partial r} - rV = 0$$

where λ representing the so-called market price of risk and $r > 0$ the risk-free interest rate.

P Kútik, K Mikula - Discrete and Continuous Dynamical Systems, 2015

Two-dimensional parabolic Black-Scholes equation

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma_1^2 {S_1}^2 \frac{\partial^2 V}{\partial {S_1}^2} + \frac{1}{2}\sigma_2^2 {S_2}^2 \frac{\partial^2 V}{\partial {S_2}^2} + \frac{1}{2}\rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + rS_1 \frac{\partial V}{\partial S_1} + rS_2 \frac{\partial V}{\partial S_2} - rV \quad (1)$$

Where $\tau = T - t$ and $(s, r, \tau) \in \Omega \times\ (0, T]$ and $u = u(S_1, S_2, \tau)$ is the three-variate u function.

One of the variables is τ and the other two variables are the spatial variables. By the two dimensional Black-Scholes equation we mean that there are two special variables in this equation.

Method for the Two-Dimensional Black-Scholes Equation.

$$
\begin{aligned}
Lu_{ij}^n = & \frac{\sigma_1^2}{2} \frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h_1^2} + \frac{\sigma_2^2}{2} \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{h_2^2}, \\
& + \rho\sigma_1\sigma_2 \frac{u_{i+1,j+1}^n - u_{i,j+1}^n + u_{i+1,j}^n + u_{i,j}^n}{h_1 h_2} \\
& + r\frac{u_{i+1,j}^n - u_{i-1,j}^n}{2h_1} + r\frac{u_{i,j+1}^n - u_{i,j-1}^n}{2h_2} - ru_{i,j}^n.
\end{aligned}
$$

- The two-dimensional Black-Scholes equation can be discretized as the one dimensional.
- Black-Scholes equation. To simplify the computations, the discrete operator L is defined.

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Alternating Direction Implicit Method

To obtain the Alternating Direction Implicit (ADI), the time derivation will be estimated as:

$$\frac{\partial u}{\partial t} \cong \frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \frac{u_{i,j}^{n+1} - u_{i,j}^{n+\frac{1}{2}} + u_{i,j}^{n+\frac{1}{2}} - u_{i,j}^{n}}{\Delta t} = \frac{u_{i,j}^{n+1} - u_{i,j}^{n+\frac{1}{2}}}{\Delta t} + \frac{u_{i,j}^{n+\frac{1}{2}} - u_{i,j}^{n}}{\Delta t}.$$

Now , we try to following equation to solve the ADI equation. First, we solve Equation in below :

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n,\frac{1}{2}}}{\Delta t} + \frac{u_{i,j}^{n,\frac{1}{2}} - u_{i,j}^{n}}{\Delta t} = L_x u_i^{n,\frac{1}{2}} + L_y u_i^{n+1}.$$

We will explain the way of obtaining the above mentioned equation in the following. To get from $-n$ to $-(n+1)$ to in ADI method, we first solve Equation in below:

$$\frac{u_{i,j}^{n,\frac{1}{2}} - u_{i,j}^{n}}{\Delta t} = L_x u_{i,j}^{n,\frac{1}{2}}$$

For $j = 1, ..., N_y$ and every one of them is a tridiagonal set from the N_x dimension. By this action, the values of u will be obtained in $n, \frac{1}{2}$ stage.

Then, to obtain u in $(n+1)$ phase, we solve:

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n,\frac{1}{2}}}{\Delta t} = L_y u_{i,j}^{n+1}$$

Then, we get to the below equation:

$$\alpha_i u_{i-1,j}^{n,\frac{1}{2}} + \beta_i u_{i,j}^{n,\frac{1}{2}} + \gamma_i u_{i+1,j}^{n,\frac{1}{2}} = f_{i,j}$$

For constant j and for $i = 1, 2, ..., N_x$ in the above equation, the equation set is defined which is a traditional set.

$$A_x = \begin{bmatrix} \beta_1 & \gamma_1 & 0 & \dots & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & \dots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \alpha_{N_x=1} & \beta_{N_x=1} & \gamma_{N_x=1} \\ 0 & 0 & 0 & \dots & \alpha_N & \beta_N \end{bmatrix}$$

The sentence $\alpha_1 u_{0,j}^{n+\frac{1}{2}}$ In the first line and the sentence $\gamma_{N_x} u_{N_{x+1}}^{n,\frac{1}{2}}$. In the last line will be transferred to right and will be subtracted from $f_{N_{x,j}}, f_{1,j}$ respectively. These sentences will be substituted from the boundary condition and the left side of the rectangular area of $\Omega = [0, L] \times [0, M]$.

$$u_{i,j}^{n,\frac{1}{2}}, u_{i-1,j}^{n,\frac{1}{2}}, u_{i+1,j}^{n,\frac{1}{2}}$$

In The present equations, the coefficient $u_{i-1,j}^{n,\frac{1}{2}}$ is :

$$\alpha_i = -\left(\frac{\sigma_1^2}{4}\frac{\Delta t}{h^2}\right)$$

And coefficient $u_{i,j}^{n,\frac{1}{2}}$ is :

$$\beta_i = \left(1 + \frac{\sigma_1^2}{2}\frac{\Delta t}{h^2} - \frac{1}{2}\frac{\Delta t}{h}r + \frac{1}{2}r\right)$$

And coefficient $u_{i+1,j}^{n,\frac{1}{2}}$ is :

$$\gamma_i = -\left(\frac{\sigma_1^2}{4}\frac{\Delta t}{h^2} + \frac{r}{2}\frac{\Delta t}{h}\right)$$

If we transfer the values in the $-n$ step to the right side and define:

$$f_{ij} := u^n_{i,j} + \frac{\Delta t}{4}\,\sigma_2^2 \frac{u^n_{i,j+1} - 2u^n_{i,j} + u^n_{i,j-1}}{h^2} + \frac{\Delta t}{2} r \frac{u^n_{i,j+1} - u^n_{i,j}}{h}$$
$$+ \frac{\Delta t}{2}\rho\sigma_1\sigma_2 \frac{u^n_{i+1,j+1} + u^n_{i-1,j-1} - u^n_{i-1,j+1} - u^n_{i+1,j-1}}{4h^2}.$$

The algorithm of this half-step is

for $j = 1 : N_y$,

for $i = 1 : N_x$,
Set $\alpha_i, \beta_i, \gamma_i$ and $f_{i,j}$

end

solve $A_x u^{n,\frac{1}{2}}_{L:N^2_{i,j}}$ by using Thomas algorithm.

end

which is a traditional set. The sentence $\alpha_1 u^{n+1}_{i,0}$ is first line and the sentence $\gamma_{N_y} u^{n+1}_{i,N_y+1}$, in the last line will be transferred to the right of the set and their values will be substituted considering the border condition of the sides in top and bottom of $[0, L] \times [0, M]$ of the rectangle area.

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ADE method for 1D Black-scholes equation

We denote by h and k the constant step sizes in the x and t directions, respectively. Let us first consider the explicit Euler method to approximate the solution of :

(a) $\frac{u_j^{n+1}-u_j^n}{k} = a^2 \frac{\partial^2 x}{\partial y^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right), 1 \leq j \leq J-1, \quad n \geq 0$

(b) $u_j^0 = f(jh), \quad 0 \leq j \leq J$

(c) $u_0^{n+1} = A, \quad u_J^{n+1} = B, \quad n \geq 0.$

The truncation error for each sweep is $O(k/h)$, but when we take the average of the two sweep solutions this error becomes $O(k^2/h)$.

In order to obtain an efficient scheme while keeping the other desirable properties, this operator is split additive by the matrix decomposition $A = L + D + U$, where L is lower diagonal, D is diagonal and U denotes an upper-diagonal matrix. We further define the symmetric splitting

$$B = L + \frac{1}{2}D \qquad C = U + \frac{1}{2}D.$$

Then we can formulate the three steps of the ADE scheme with its upward/downward sweeps and the combination (also for higher dimensions) as:

$$\begin{aligned} \text{UP } u^{n+1} &= \left[I - kB\left(v^n\right)\right]^{-1}\left[I + kC\left(v^n\right)\right]v^n, \\ \text{DOWN } d^{n+1} &= \left[I - kC\left(v^n\right)\right]^{-1}\left[I + kB\left(v^n\right)\right]v^n, \\ \text{COMB } v^{n+1} &= \frac{1}{2}\left[u^{n+1} + d^{n+1}\right]. \end{aligned}$$

ADE method for Two Dimensional Black-Sholes equation

Fichera Conditions
For solving 2D Black-Scholes equation, we considering that $\tau = T - t$, and $V = V(S, r, \tau)$ (r is VaŠiček interest rate).
We will have the following equations

$$\frac{\partial V}{\partial \tau} + \overset{\rightarrow}{\alpha} . \nabla V - \nabla . \tilde{B} \nabla V + rV = 0$$

Where

$$\overset{\rightarrow}{\alpha} = - \begin{pmatrix} rS - vS - \frac{1}{2}\rho\sigma S \\ \kappa(\theta - v) - \lambda v - \frac{1}{2}\sigma^2 - \frac{1}{2}\rho\sigma v \end{pmatrix}.$$

and

$$\tilde{B} = \begin{pmatrix} \frac{1}{2}\sigma S^2 & \frac{1}{2}\rho\sigma S \\ \frac{1}{2}\rho\sigma S & \frac{1}{2}\sigma^2 \end{pmatrix}$$

We represent the solution that is an estimation,

$$u^n_{r,s} \approx u(x_r, y_s, t_n), \quad 0 \leq r \leq J_x, \; 0 \leq S \leq J_x, \; 0 \leq n \leq N.$$

We will frequently utilize the second-order difference operator σ_x^2 in the following discussion. This operator is defined as

$$\sigma_x^2 u_{r,s}^n = \frac{u_{r+1,s}^n - 2u_{r,s}^n + u_{r-1,s}^n}{(\Delta x)^2}.$$

By using the σ_x^2, we can obtain an explicit scheme for Black-Sholes equation.

$$\frac{u_{r,s}^{n+1} - u_{r,s}^n}{\Delta h} = \frac{\sigma_x^2 u_{r,s}^{n+1}}{(\Delta x)^2} - \frac{\sigma_y^2 u_{r,s}^{n+\frac{1}{2}}}{(\Delta y)^2}. \tag{2}$$

Then, we have:

$$\frac{u_{r,s}^{n+1} - u_{r,s}^{n}}{\Delta t} = \frac{1}{2}\left[\frac{\sigma_x^2\left(u_{r,s}^{n+1} - u_{r,s}^{n+\frac{1}{2}}\right)}{(\Delta x)^2} + \frac{\sigma_y^2\left(u_{r,s}^{n+\frac{1}{2}} - u_{r,s}^{n}\right)}{(\Delta y)^2}\right].$$

Denote $\mu_x = \frac{\Delta t}{(\Delta x)^2},\ \mu_y = \frac{\Delta t}{(\Delta y)^2}$.

This equation has the potential to be expressed in a different manner

$$\left(1 - \frac{1}{2}\mu_x\sigma_x^2 - \frac{1}{2}\mu_y\sigma_y^2\right)u_{r,s}^{n+1} = \left(1 + \frac{1}{2}\mu_x\sigma_x^2 + \frac{1}{2}\mu_y\sigma_y^2\right)u_{r,s}^{n}.$$

In terms of stability, it is readily achievable to determine the amplification factor

$$\lambda(k) = \frac{1 - 2\mu_x \sin^2(\frac{1}{2}k_x)\Delta x - 2\mu_y \sin^2(\frac{1}{2}k_y)\Delta y}{1 + 2\mu_x \sin^2(\frac{1}{2}k_x)\Delta x + 2\mu_y \sin^2(\frac{1}{2}k_y)\Delta y},$$

whose amplitude is always less than or equal to one for any mesh sizes Δx and Δy. Hence, the ADE method is unconditionally stable.

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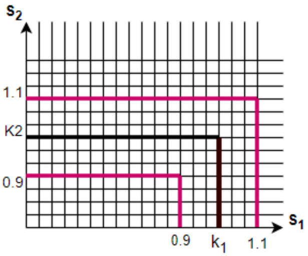
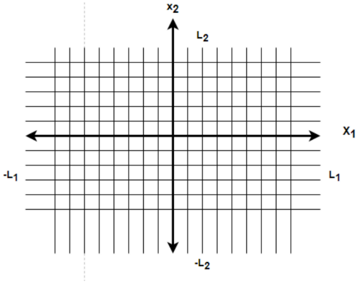
All cash or none

First, the option of two cash assets or none will be considered. We assume that by having two assets x, y the income of option is as follows:



(a) Original Variables



(b) Transformed variables

$$X_i = \log \frac{S_i}{K_i} \quad i = 1, 2$$

Figure 1: G=$\{(S_1, S_2) \mid 0.9K_1 \leq S_1 \leq 1.1K_1, \;\; 0.9K_2 \leq S_2 \leq 1.1K_2\}$

$$\Lambda(x,y) = \begin{cases} \text{cash} & S_1 \geq K_1, \;\; S_2 \geq K_2 \\ 0 & \text{unless} \end{cases}$$

where $K_1$and K_2 are the prices of S_1, S_2. The function figure is as follows.
The following values will be used for the numerical simulation of the parameters.

$$\sigma_1 = \sigma_2 = 0.3, \;\; r = 0.03, \;\; \rho = 0.5, \;\; cash = 1, \;\; K_1 = K_2 = 100,$$

We consider the calculation domain as $\Omega = [0, 300] \times [0, 300]$.

Option to buy on a maximum of Two assets

In this example, the option to purchase a maximum of two cash assets is considered. Income of purchase option are considered as:

$$¥(x,y) = max(S_1 - K_1, \;\; S_2 - K_2, 0)$$

Which are X_1 and X_2 futures prices and x and y current asset prices are the first and second, respectively.

$M = [0, 300], \; L = [0, 300].$
In this case the Dirichlet condition in $y = M, x = L$ but in $y = 0, \;\; x = 0$ the boundary linear condition is used.

Conclusion and outlook

- We presented the idea of ADI and ADE methods.
- Our aim is to compare the two numerical methods for solving 2D Black-Scholes equations.
- We evaluate the performance of the two approaches based on some assumptions.

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Thanks for your attention!