A fast Monte Carlo scheme for additive processes and option pricing

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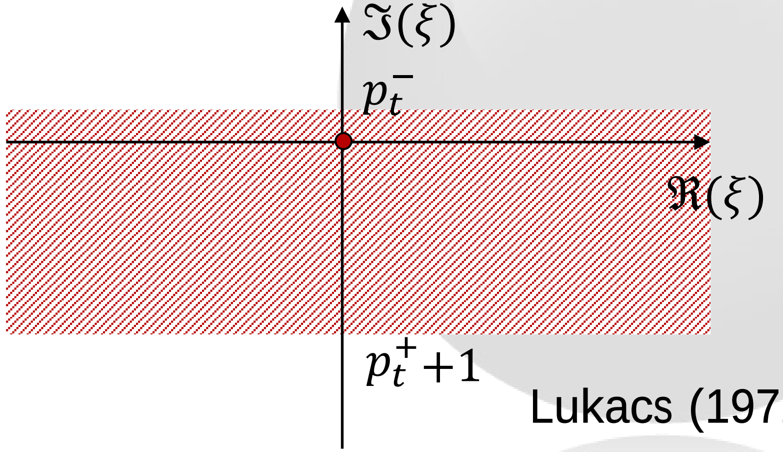
Basic vocabulary

- ✓ Additive process — Time-inhomogeneous Lèvy processes

- ✓ characteristic function — $\phi_t(\xi) := E\left[e^{i\, X_t\, \xi}\right]$

- ✓ characteristic function for increments — $\phi_{s,t}(u) = \dfrac{\phi_t(u)}{\phi_s(u)}$

- ✓ bounds of analyticity strip for $\phi_t(\xi)$



Lukacs (1972)

We simulate increments for path-dependent products

Additive processes in finance

Applications in quantitative finance are relatively few:

- Carr-Geman-Madan-Yor (2007) introduce self-similar (or Sato) processes in derivative pricing;

- Li-Li-MendozaArriaga (2016) build a quite large class of Additive proc. via Levy subordination;

… but are the new frontier:

- Madan-Wang (2020, 2023) Additive bilateral VG and time embedding.

- Carr-Torricelli (2021), European options (Black & Bachelier like)
  with a simple closed formula;

- A.-Baviera (2021, 2022) Excellent calibration properties and power law scaling.

Jump based approaches:

- Cont, R. & Tankov, P. (2003). And references therein (e.g. for Gaussian-Aproximation)
- Eberlein, E. & Madan, D. B. (2009). Pricing of structured products.

CDF-FFT based approaches for Lèvy:

- Glasserman-Liu (2010). Error bounds estimation due to linear CDF interpolation.
- Chen-Feng-Lin (2012). Simulation via Sinc method;
- Ballotta-Kyriakou (2014) An FFT simulation technique with error bounds estimation.

The idea: Simulation Lewis-FFT-S method

The method:

➤ CDF *via*

- **Lewis**

$$P(x) = R_a - \frac{e^{-a\,x}}{2\,\pi} \int_{-\infty}^{\infty} du\, \frac{e^{-i\,u\,x}\,\phi_{s,t}(u-ia)}{i\,u+a}$$

with R_a a constant $\qquad a \in (0, p_t^+ + 1)$

- **FFT**

$$\hat{P}(x) := 1 - \frac{e^{-ax}}{\pi} \sum_{l=0}^{N-1} Re\left[\frac{e^{-i(l+1/2)hx}\phi_{s,t}((l+1/2)h - i\,a)}{i\,(l+1/2)h+a}\right]$$

Lee (2004)

➤ r.v. drawn *via* Spline interpolation

Our contributions:

- ✓ Estending to additive processes
- ✓ Selecting optimal a
- ✓ Spline interpolation

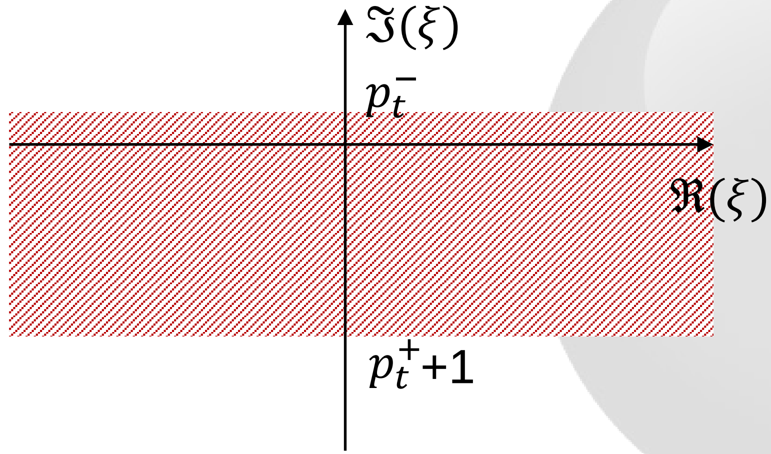
Chen-Feng-Lin (2012): a=0

➢ **Theorem 2.1**

p_t^+ & p_t^- (bounds analyticity strip) non increasing in t for all additive processes.

Consequence: to simulate the increment we always consider p_t^+ & p_t^-

$$\phi_{s,t} = \frac{\phi_t(u)}{\phi_s(u)}$$



Theoretical results

- Assumption: Exponential Decay: $|\phi_{s,t}(u - i\,a)| < Be^{-b\,u^{\omega}}$ $\quad \forall a \in (0, p_t^+ + 1)$

- **Proposition 2.2.**

  If the Assumption hold

  1. CDF error bound (N number of grid points)

  $$\mathcal{E}_N^{CDF}(x) = O(N^{-\omega/(1+\omega)})\boxed{\exp(-bN^{\omega/(1+\omega)})} \qquad \omega > 0$$

  2. optimal bound for $a = (p_t^+ + 1)/2$

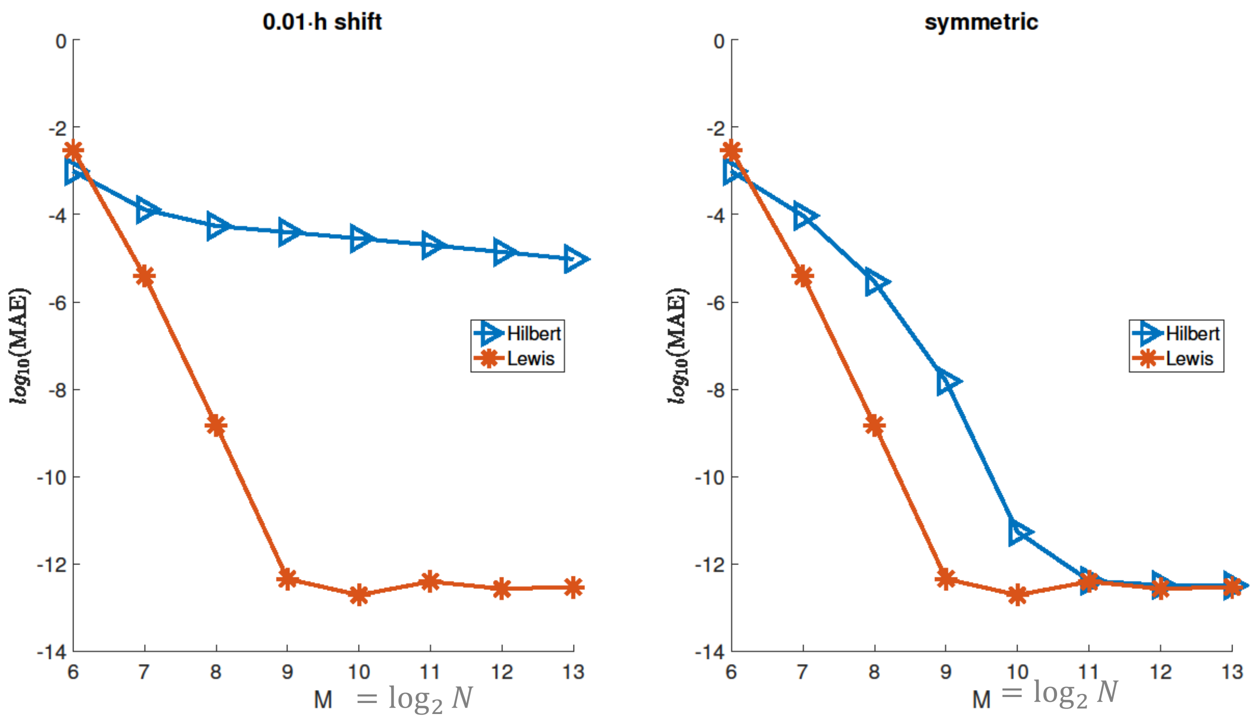
It is possible to have an error bound also for power decay.

Comparison with Hilbert Transform

Error bounds:

Chen-Feng-Lin (2012): $\sim \max\left(e^{\frac{\pi p_t^-}{h}}, e^{-\frac{\pi(p_t^+ +1)}{h}}\right)$

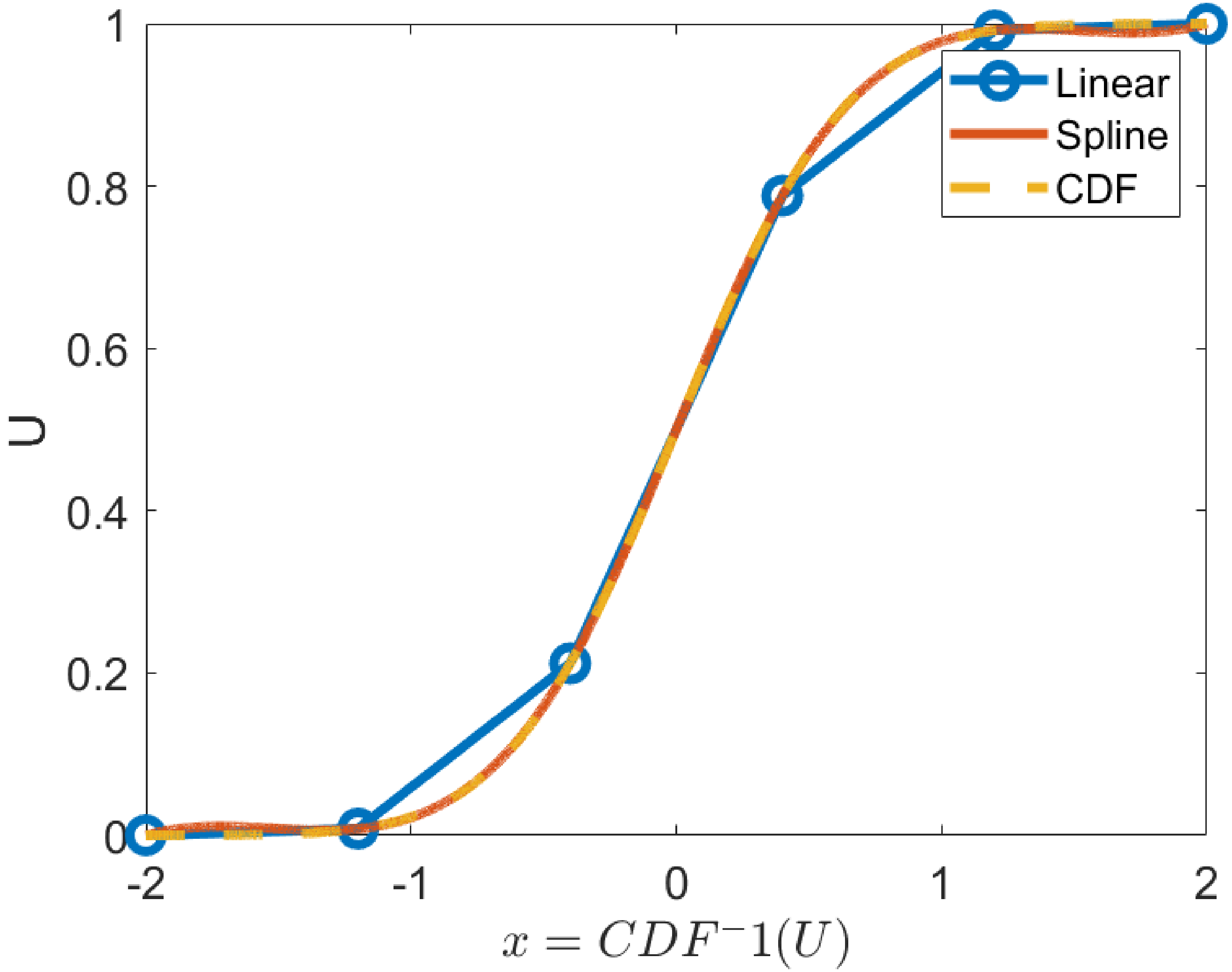
Lewis with optimal a: $\sim \min\left(e^{\frac{\pi p_t^-}{h}}, e^{-\frac{\pi(p_t^+ +1)}{h}}\right)$



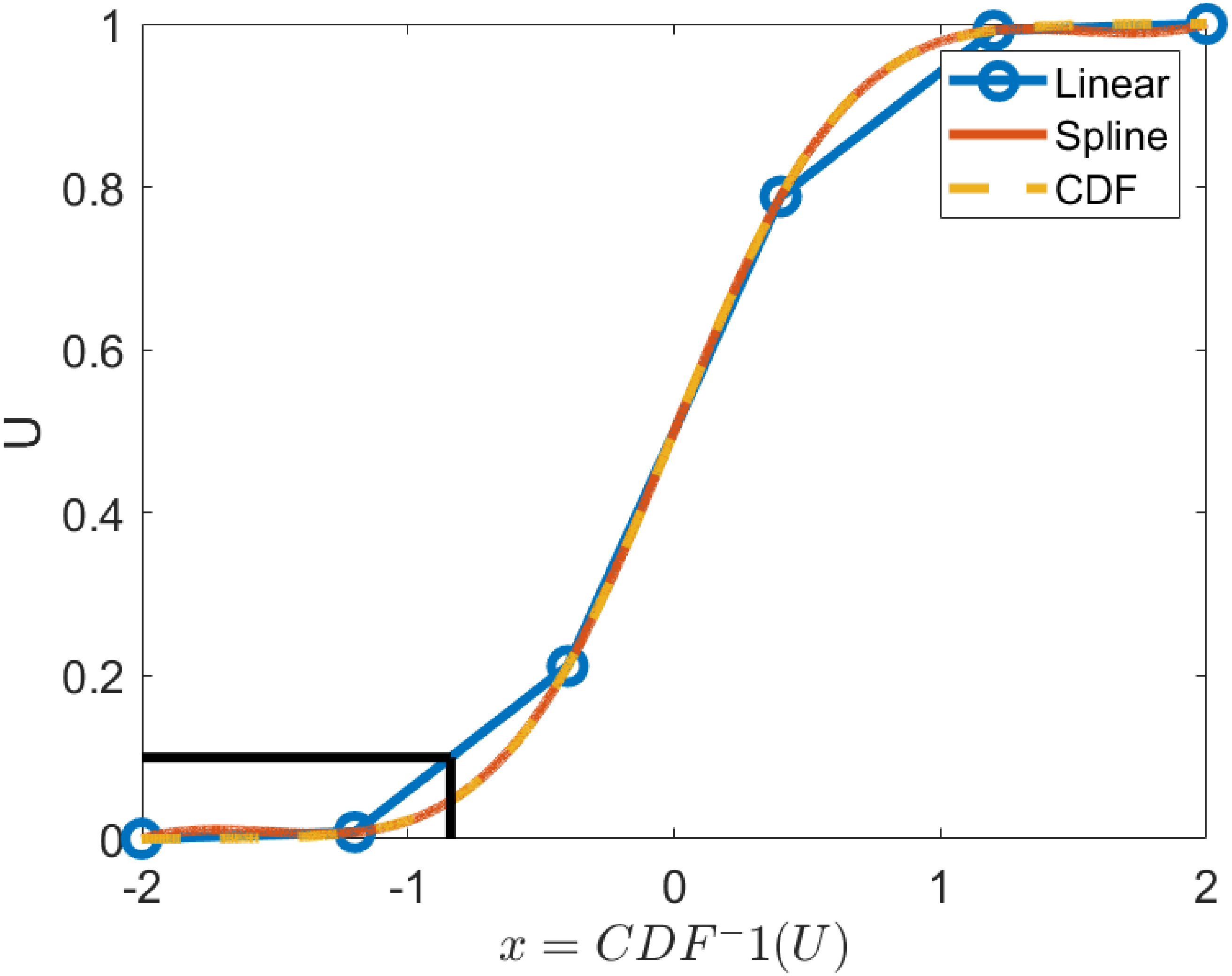
CDF inversion error with and without a symmetric grid

Unstable for numerical routines

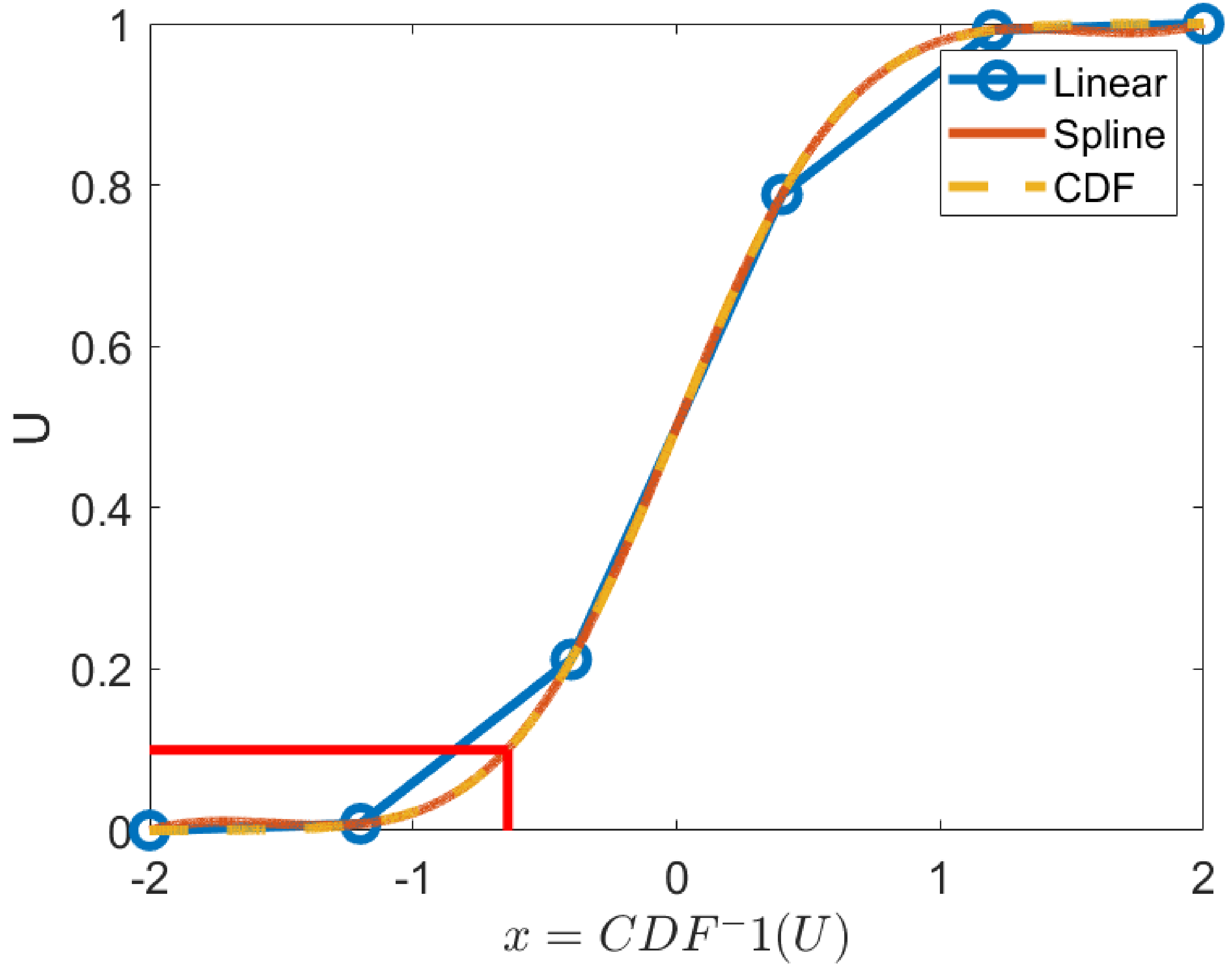
Sampling from approximated CDF



Sampling from approximated CDF



Sampling from approximated CDF



Sampling from approximated CDF (linear interpolations)

Approximated CDF $\hat{P}$ on a grid $x_1, \dots x_j, \dots x_K$
Generate N_{Sim} uniform random variables *U*

Steps

1. Select j *s.t.* $\hat{P}(x_{j-1}) \leq U \leq \hat{P}(x_j)$ *(N-N alg.)*
2. Determine the linear interpolation coefficients c_0^j and c_1^j
3. Compute $X = c_0^j + c_1^j U$

Comp. Costs

1. $N_{sim} \log_2 N_{Sim}$
2. $6\ K$
3. N_{sim}

Sampling from approximated CDF (Spline interpolation)

In practice *K* (grid size) << N_{sim} → Similar computational times for spline and linear interpolation

Steps

1. Select j *s.t.* $\hat{P}(x_{j-1}) \leq U \leq \hat{P}(x_j)$
2. Determine the spline interpolation coefficients
3. Compute $X = spline(U)$

Comp. Costs

1. $N_{sim} \log_2 N_{sim}$
2. $8K - 7$
3. $\approx N_{sim}$

Error contributions

When pricing a generic derivative with an approximated CDF

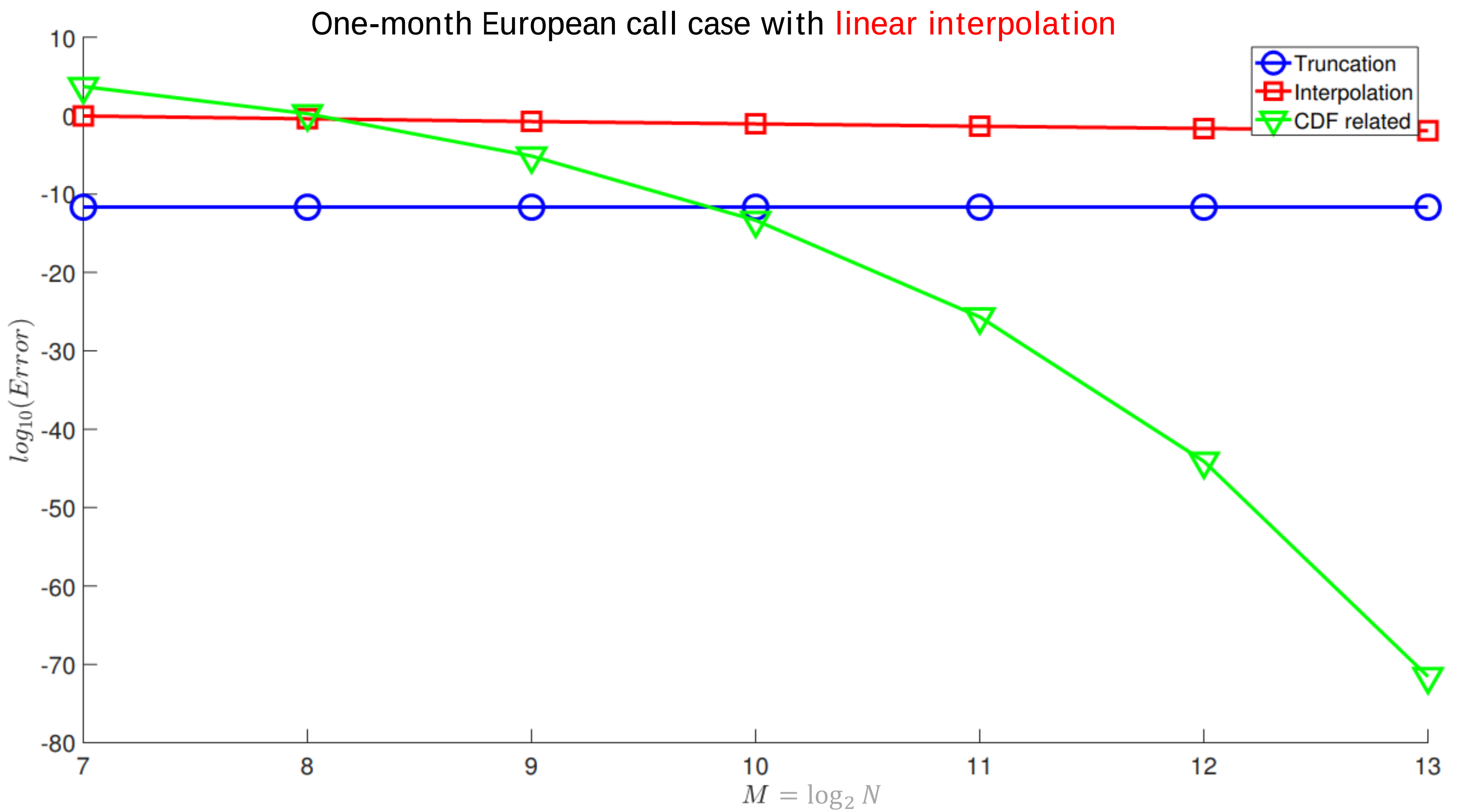
Error < CDF related Error +

Interpolation Error +

Truncation Error

Upper bounds are analytical

Error contributions



Upper bounds:

Linear interpolation: $O\left(\frac{1}{K^2}\right)$

Spline interpolation: $O\left(\frac{1}{K^6}\right)$

The model: Additive Tempered Stable (ATS)

Forward with expiry T is modeled as an exponential Additive

$$F_t(T) \coloneqq F_0(T)\exp(f_t)$$

with f_t an ATS, whose **characteristic function** is

$$\mathbb{E}\left[e^{iuf_t}\right] = \mathcal{L}_t\left(iu\left(\frac{1}{2}+\eta_t\right)\sigma_t{}^2 + \frac{u^2\sigma_t^2}{2};\ k_t,\ \alpha\right) e^{-iu\,t\ \ln\mathcal{L}_t\left(\sigma_t^2\,\eta_t;\,k_t\right)}$$

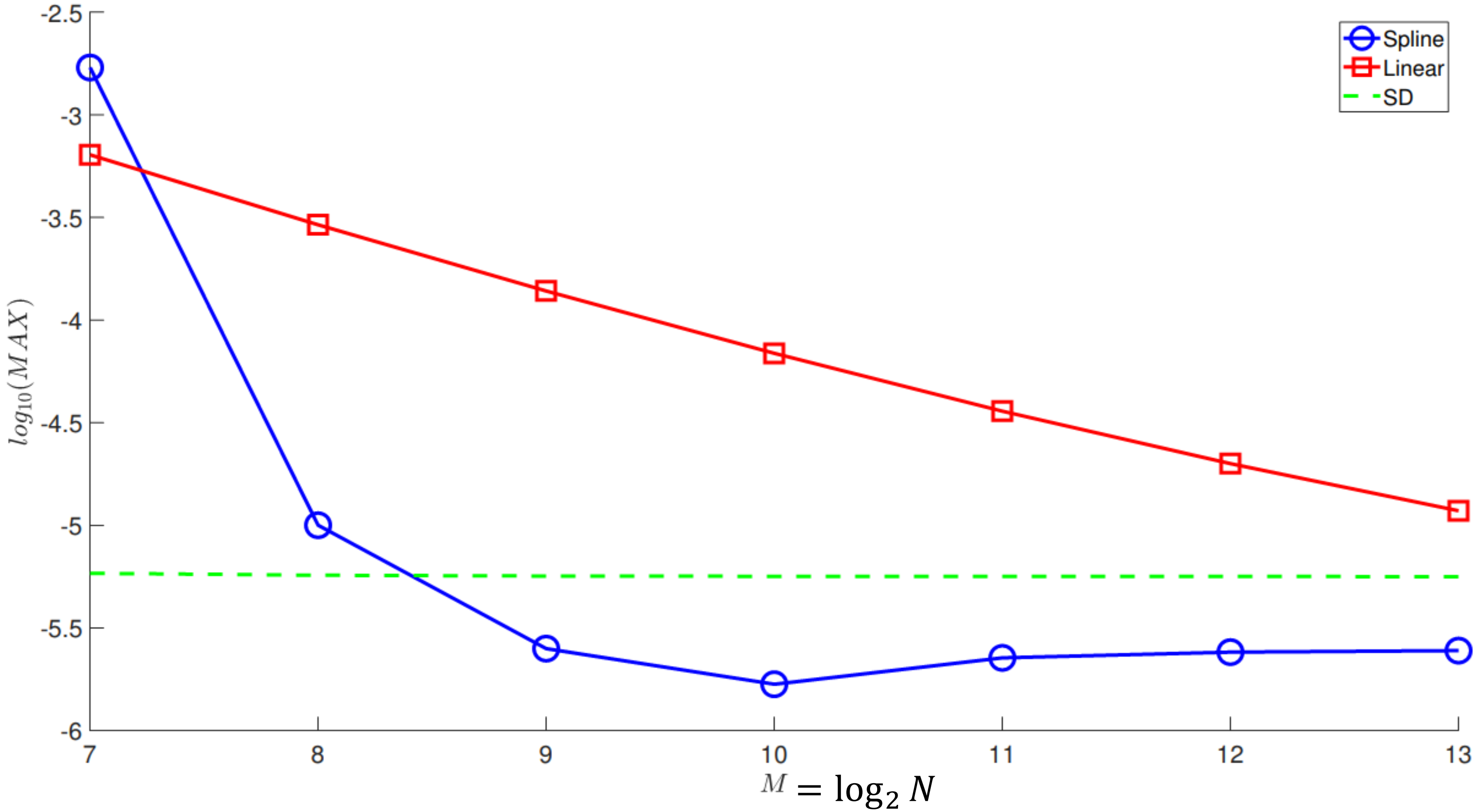
with $\quad k_t = \bar{k}t^{\beta} \quad \eta_t = \bar{\eta}t^{\delta} \quad \sigma_t = \sigma \quad \alpha \in (0,1)$

Excellent calibration

Match IV skew

Skew: slope of the IV ATM. Risk management of derivatives

Interpolation error: linear *vs* spline



Bias estimation: MAX error on 30 calls with moneyness degree in the range $(-0.2, 0.2)$
'Variance' estimation: SD error with 10^7 simulations

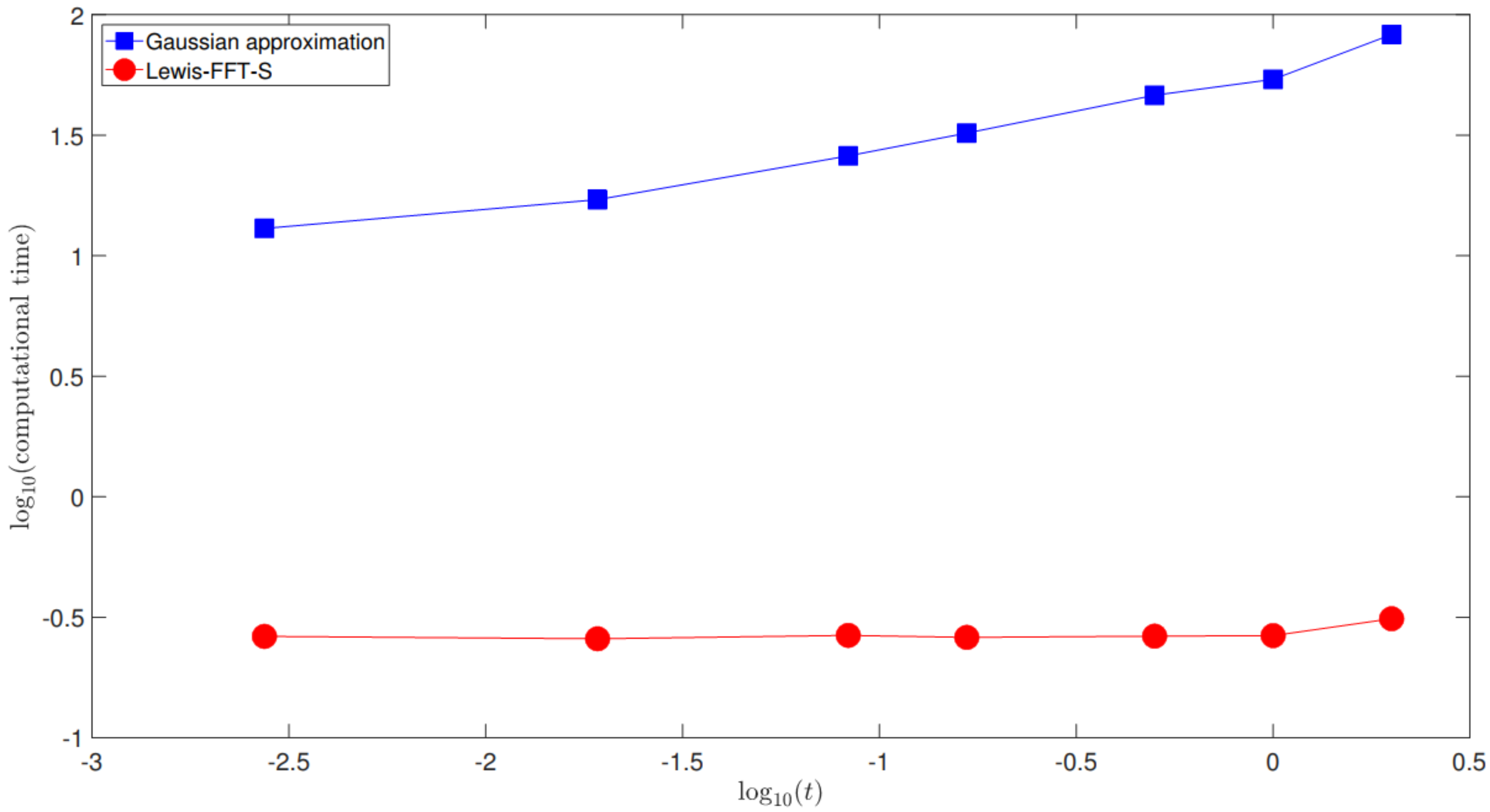
- Plain Vanilla calls (30 calls, $ttm = 1m$, moneyness degree in the range $(-0.2, 0.2)$, $N_{sim} = 10^7$)

	M	7	8	9	10	11	12	13
$\alpha = 1/3$	MAX [bp]	7.08	0.32	0.02	0.03	0.03	0.03	0.03
	RMSE [bp]	3.49	0.29	0.01	0.02	0.02	0.02	0.02
	MAPE [%]	2.30	0.21	0.01	0.01	0.01	0.01	0.01
	SD [bp]	0.12	0.12	0.12	0.12	0.12	0.12	0.12
$\alpha = 2/3$	MAX [bp]	317.29	0.19	0.04	0.01	0.02	0.03	0.03
	RMSE [bp]	282.99	0.16	0.03	0.01	0.02	0.02	0.02
	MAPE [%]	185.64	0.11	0.02	0.01	0.01	0.01	0.01
	SD [bp]	0.25	0.11	0.11	0.11	0.11	0.11	0.11

- Discretely monitored options (5y, Q/Q)

Moneyness	Asian [%]	SD [%]	Lookback [%]	SD [%]	Down-and-In [%]	SD [%]
-0.5	39.79	0.01	3.31	0.00	2.31	0.00
-0.25	24.36	0.01	8.72	0.00	3.98	0.00
0	10.04	0.01	23.07	0.01	6.15	0.01
0.25	2.57	0.00	50.53	0.01	8.95	0.01
0.5	0.55	0.00	86.98	0.01	12.55	0.01

Computational time: Lewis-FFT-S vs GA (with Ziggurat)



with Lewis-FFT-S error $\leq_{\text{always}}$ GA error

Computational time: Lewis-FFT-S vs GBM

Simulation 10^7 trials

	M	7	8	9	10	11	12	13
$\alpha = 1/3$	Time [s]	0.23	0.27	0.28	0.28	0.28	0.28	0.29
$\alpha = 2/3$	Time [s]	0.25	0.27	0.28	0.28	0.28	0.28	0.28

$\simeq$ *3 times GBM*

Conclusions on Lewis-FFT-S

- A new MC technique for Additive processes based on FFT
- We improve the two main sources of numerical errors (CDF inversion and linear interpolation)
- Very fast: same order of magnitudes as GBM.

Conclusions: additive vs other model classes. Not only fast-simulation!

	Additive processes		Stochastic volatility		Rough volatility	
simple derivatives	closed formula	✓	closed formula	✓	NO closed formula	✗
short time	$skew \sim t^{-\frac{1}{2}}$	✓	$skew \sim const(t)$	✗	$skew \sim t^{-\frac{1}{2}}$	✓
calibration	• skew	✓	• wrong skew	✗	• skew	✓
	• very low MSE/MAPE	✓	• high MSE/MAPE	✗	• (?) MSE/MAPE	✓ ✗
simulation	fast $\sim O(1)\ GBM$	✓	slow $\sim 10^2 - 10^3\ GBM$	✗	very slow $\sim (\ ?\ )\ GBM$	✗

Bibliography sketch (I)

[1] Azzone, M. & Baviera, R. (2021). *Additive normal tempered stable processes for equity derivatives and power law scaling*. Quantitative Finance, 1-18.

[2] Azzone, M. & Baviera, R. (2022). *Short-time implied volatility of additive normal tempered stable processes* Annals of operations research, 1-34

[3] Ballotta, L. & Kyriakou, I. (2014). *Monte Carlo simulation of the CGMY process and option pricing*. Journal of Futures Markets, 34(12), 1095-1121.

[4] Benth, F.E. & Sgarra, C. (2012). *The risk premium and the Esscher transform in power markets*, Stochastic Analysis and Applications, 30 (1), 20-43.

[5] Carr, P., Geman, H., Madan, D.B. & Yor, M. (2007). *Self-decomposability and option pricing*, Mathematical Finance, 17 (1), 31-57

[6] Carr, P. & Torricelli, L. (2021). *Additive logistic processes in option pricing*, Fin. Stoch., 25, 689–724

[7] Chen, Z., Feng, L. &Lin, X. (2012). Simulating Lévy processes from their characteristic functions and financial applications, ACM Transactions on Modeling and Computer Simulation, 22 (3), 1-26.

Bibliography sketch (II)

[8] Cont, R. & Tankov, P. (2003). *Financial Modelling with jump processes*, Chapman Fin. Math. Series

[9] Eberlein, E. & Madan, D. B. (2009). Sato processes and the valuation of structured products. Quantitative Finance, 9(1), 27-42.

[10] Glasserman, P. & Liu, Z. (2010). *Sensitivity estimates from characteristic functions*, Operations Research, 58 (6), 1611-1623.

[11] Lee R. (2004). Option Pricing by Transform Methods: Extensions, Unification, and Error Control, Journal of Computational Finance, 7(3):51–86

[12] Li, J., Li, L. & Mendoza-Arriaga, R. (2016). *Additive subordination and its applications in finance*, Finance and Stochastics, 20 (3), 589-634

[13] Madan, D. B., & Wang, K. (2020). *Additive processes with bilateral Gamma marginals*. Applied Mathematical Finance, 27(3), 171-188.

The paper

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