# Pricing American and Bermudan Options using Almost-Exact Scheme of Heston-type Models

Mara Kalicanin Dimitrov[1], Marko Dimitrov[1], Anatoliy Malyarenko[1], Ying Ni[1]

[1]Division of Mathematics and Physics,
Mälardalen University

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# Introduction and Motivation

- American and Bermudan Options;
- Heston-type Models are more realistic than the classical Black-Scholes;
- Almost-Exact Scheme;
- Small Number of Steps.

# Heston Model

The Heston model is given by a system of two stochastic differential equations:

$$dS_t = rS_t dt + \sqrt{\nu_t} S_t dW_t^1,$$
$$d\nu_t = \kappa\left(\bar{\nu} - \nu_t\right) dt + \gamma\sqrt{\nu_t} dW_t^2$$

where $r$ is a constant risk-free interest rate, $\kappa > 0$ is called the speed of mean reversion rate for the variance. Parameter $\bar{\nu} > 0$ is the long-term mean of the variance process, and $\gamma > 0$ is the volatility of the variance. The initial level of the variance is given by $v_0 > 0$. Wiener processes $W_t^1$ and $W_t^2$ are correlated with correlation coefficient $\rho_{12} \in [-1, 1]$.

# Double Heston Model

The Double Heston model is given by a system of three stochastic differential equations:

$$dS_t = rS_t dt + \sqrt{\nu_t^1} S_t dW_t^1 + \sqrt{\nu_t^2} S_t dW_t^2,$$
$$d\nu_t^1 = \kappa_1 \left(\bar{\nu}_1 - \nu_t^1\right) dt + \gamma_{v_1} \sqrt{\nu_t^1} dW_t^3,$$
$$d\nu_t^2 = \kappa_2 \left(\bar{\nu}_2 - \nu_t^2\right) dt + \gamma_{v_2} \sqrt{\nu_t^2} dW_t^4$$

where the Wiener processes $W_t^1$ and $W_t^3$ are correlated with correlation coefficient $\rho_{1,3}$, $W_t^2$ and $W_t^4$ are correlated with correlation coefficient $\rho_{2,4}$.

The Feller condition is given by

$$2\kappa_i\theta_i > \sigma_{v_i}^2, \text{ for } i = 1, 2.$$

The Feller condition guarantees that the process $v_t^i$ are positive.

# AES of the Heston Model

The Almost Exact Scheme (AES) under the Heston model is suggested in Oosterlee and Grzelak book [1].

The AES of the Heston model, where $X_t := \ln S_t$, is given by

$$x_{i+1} \approx x_i + c_0 + c_1 v_i + c_2 v_{i+1} + \sqrt{c_3 \cdot v_i} Z_1,$$

and the CIR process is simulated using

$$v_{i+1} \approx \bar{c}(t_{i+1}, t_i) \cdot \chi^2 \left(\delta, \bar{\kappa}\left(t_{i+1}, t_i\right)\right),$$

where the parameters are given by

$$\bar{c}\left(t_{i+1}, t_i\right) = \frac{\gamma^2}{4\kappa}\left(1 - \mathrm{e}^{-\kappa\left(t_{i+1}-t_i\right)}\right), \quad \bar{\kappa}\left(t_{i+1}, t_i\right) = \frac{4\kappa \mathrm{e}^{-\kappa\left(t_{i+1}-t_i\right)}}{\gamma^2\left(1 - \mathrm{e}^{-\kappa\left(t_{i+1}-t_i\right)}\right)} \cdot v_i,$$

and $\chi^2(\delta, \bar{k}(\cdot,\cdot))$ represents the noncentral chi-squared distribution with degrees of freedom and noncentrality parameter $\bar{k}(\cdot,\cdot)$. The parameters are given by

$$c_0 = \left(r - \frac{\rho_{1,2}}{\gamma}\kappa\bar{\nu}\right)\Delta t, \quad c_1 = \left(\frac{\rho_{1,2}}{\gamma}\kappa - \frac{1}{2}\right)\Delta t - \frac{\rho_{1,2}}{\gamma}, \quad c_3 = \left(1 - \rho_{1,2}^2\right)\Delta t.$$

# AES of the Double Heston Model

By applying Itô's lemma to the log transformation $X_t := \ln S_t$ and the Cholesky decomposition of the correlation matrix, we have the following

$$
\begin{aligned}
dX_t &= \left(r - \frac{1}{2}\left(\nu_t^1 + \nu_t^2\right)\right) dt + \sqrt{\nu_t^1}\left(\rho_{1,3} d\widetilde{W}_t^3 + \sqrt{1-\rho_{1,3}^2}\, d\widetilde{W}_t^1\right) \\
&\quad + \sqrt{\nu_t^2}\left(\rho_{2,4} d\widetilde{W}_t^4 + \sqrt{1-\rho_{2,4}^2}\, d\widetilde{W}_t^2\right), \\
d\nu_t^1 &= \kappa_1\left(\bar{\nu}_1 - \nu_t^1\right) dt + \gamma_{\nu_1}\sqrt{\nu_t^1}\, d\widetilde{W}_t^3, \\
d\nu_t^2 &= \kappa_2\left(\bar{\nu}_2 - \nu_t^2\right) dt + \gamma_{\nu_2}\sqrt{\nu_t^2}\, d\widetilde{W}_t^4,
\end{aligned}
$$

where $d\widetilde{W}_t^1, d\widetilde{W}_t^2, d\widetilde{W}_t^3$ and $d\widetilde{W}_t^4$ are independent Brownian increments.

Taking integration of all processes in a certain time interval $[t_i, t_{i+1}]$, we get the following discretization scheme

$$\begin{aligned} x_{i+1} =& x_i + \int_{t_i}^{t_{i+1}} \left( r - \frac{1}{2}\left(v_t^1 + v_t^2\right)\right) \mathrm{d}t + \rho_{1,3} \int_{t_i}^{t_{i+1}} \sqrt{v_t^1} \mathrm{d}\widetilde{W}_t^3 + \sqrt{1-\rho_{1,3}^2} \int_{t_i}^{t_{i+1}} \sqrt{v_t^1} \mathrm{d}\widetilde{W}_t^1 \\ & + \rho_{2,4} \int_{t_i}^{t_{i+1}} \sqrt{v_t^2} \mathrm{d}\widetilde{W}_t^4 + \sqrt{1-\rho_{2,4}^2} \int_{t_i}^{t_{i+1}} \sqrt{v_t^2} \mathrm{d}\widetilde{W}_t^2, \end{aligned} \tag{1}$$

$$v_{i+1}^1 = v_i^1 + \kappa_1 \int_{t_i}^{t_{i+1}} \left(\theta_1 - v_t^1\right) \mathrm{d}t + \sigma_{v_1} \int_{t_i}^{t_{i+1}} \sqrt{v_t^1} \mathrm{d}\widetilde{W}_t^3,$$

$$v_{i+1}^2 = v_i^2 + \kappa_2 \int_{t_i}^{t_{i+1}} \left(\theta_1 - v_t^2\right) \mathrm{d}t + \sigma_{v_2} \int_{t_i}^{t_{i+1}} \sqrt{v_t^2} \mathrm{d}\widetilde{W}_t^4.$$

Now we can notice that there are integrals with $\widetilde{W}_t^i, i = 3, 4$ in the SDEs above that are the same. In terms of the variance realizations the following holds:

$$\begin{aligned}
\int_{t_i}^{t_{i+1}} \sqrt{v_t^1} \mathrm{d}\widetilde{W}_t^3 &= \frac{1}{\sigma_{v_1}} \left( v_{i+1}^1 - v_i^1 - \kappa_1 \int_{t_i}^{t_{i+1}} \left( \theta_1 - v_t^1 \right) \mathrm{d}t \right), \\
\int_{t_i}^{t_{i+1}} \sqrt{v_t^2} \mathrm{d}\widetilde{W}_t^4 &= \frac{1}{\sigma_{v_2}} \left( v_{i+1}^2 - v_i^2 - \kappa_2 \int_{t_i}^{t_{i+1}} \left( \theta_2 - v_t^2 \right) \mathrm{d}t \right).
\end{aligned} \tag{2}$$

For $\Delta t = t_{i+1} - t_i$ the discretization for $x_{i+1}$ is given by

$$
\begin{aligned}
x_{i+1} \approx x_i &+ \left(r - \frac{1}{2}\left(v_i^1 + v_i^2\right)\right)\Delta t + \frac{\rho_{1,3}}{\gamma_{v_1}}\left(v_{i+1}^1 - v_i^1 - \kappa_1\left(\bar{\nu}_1 - v_i^1\right)\Delta t\right) \\
&+ \sqrt{1-\rho_{1,3}^2}\sqrt{v_i^1}\left(\widetilde{W}_{t_{i+1}}^1 - \widetilde{W}_{t_i}^1\right) + \frac{\rho_{2,4}}{\gamma_{v_2}}\left(v_{i+1}^2 - v_i^2 - \kappa_2\left(\bar{\nu}_2 - v_i^2\right)\Delta t\right) \\
&+ \sqrt{1-\rho_{2,4}^2}\sqrt{v_i^2}\left(\widetilde{W}_{t_{i+1}}^2 - \widetilde{W}_{t_i}^2\right).
\end{aligned}
$$

The AES of the Double Heston model is given by

$$
x_{i+1} \approx x_i + c_0 + c_1 v_i^1 + c_2 v_i^2 + c_3 v_{i+1}^1 + c_4 v_{i+1}^2 + \sqrt{c_5 \cdot v_i^1} Z_1 + \sqrt{c_6 \cdot v_i^2} Z_2,
$$

and the two CIR processes

$$
\begin{aligned}
v_{i+1}^1 &\approx \bar{c}^1(t_{i+1}, t_i)\cdot\chi^2\left(\delta^1, \bar{\kappa}^1\left(t_{i+1}, t_i\right)\right), \\
v_{i+1}^2 &\approx \bar{c}^2(t_{i+1}, t_i)\cdot\chi^2\left(\delta^2, \bar{\kappa}^2\left(t_{i+1}, t_i\right)\right).
\end{aligned}
$$

Previously mentioned parameters are given as follows

$$\bar{c}^j\left(t_{i+1}, t_i\right) = \frac{\sigma^2}{4\kappa}\left(1 - \mathrm{e}^{-\kappa\left(t_{i+1}-t_i\right)}\right),$$

$$\bar{\kappa}^j\left(t_{i+1}, t_i\right) = \frac{4\kappa \mathrm{e}^{-\kappa\left(t_{i+1}-t_i\right)}}{\sigma^2\left(1 - \mathrm{e}^{-\kappa\left(t_{i+1}-t_i\right)}\right)} \cdot v_i^j,$$

for $j = 1, 2$, and $\chi^2(\delta, \bar{k}(\cdot,\cdot))$ represents the noncentral chi-squared distribution with degrees of freedom and noncentrality parameter $\bar{k}(\cdot,\cdot)$. The constants are

$$c_0 = \left(r - \frac{\rho_{1,3}}{\sigma_{V_1}}\kappa_1\theta_1 - \frac{\rho_{2,4}}{\sigma_{V_2}}\kappa_2\theta_2\right)\Delta t, \qquad c_1 = \left(\frac{\rho_{1,3}}{\sigma_{V_1}}\kappa_1 - \frac{1}{2}\right)\Delta t - \frac{\rho_{1,3}}{\sigma_{V_1}},$$

$$c_2 = \left(\frac{\rho_{2,4}}{\sigma_{V_2}}\kappa_2 - \frac{1}{2}\right)\Delta t - \frac{\rho_{2,4}}{\sigma_{V_2}}, \qquad c_3 = \frac{\rho_{1,3}}{\sigma_{V_1}},$$

$$c_4 = \frac{\rho_{2,4}}{\sigma_{V_2}}, \qquad c_5 = \left(1 - \rho_{1,3}^2\right)\Delta t, \quad c_6 = \left(1 - \rho_{2,4}^2\right)\Delta t.$$

# AES of the Heston Model

- Longstaff and Schwartz[2] Least Square Monte Carlo (LSM) method;
- Global Polynomials & Laguerre Polynomials as basis functions;
- $1,000,000$ paths, 30 runs;
- In-the-money paths;
- Parameters for American option pricing using two choices of $\nu_0$: $\nu_0 = 0.0625$ and $\nu_0 = 0.25$ are given in the table below.

| $K$ | $T$ | $r$ | $\gamma$ | $\bar{\nu}$ | $\kappa$ | $\rho$ |
|---|---|---|---|---|---|---|
| 10 | 0.25 | 0.1 | 0.9 | 0.16 | 5 | 0.1 |

Table 1: Parameters for pricing American put options under Heston model.

# American Options: $\nu_0 = 0.0625$ and $\nu_0 = 0.25$

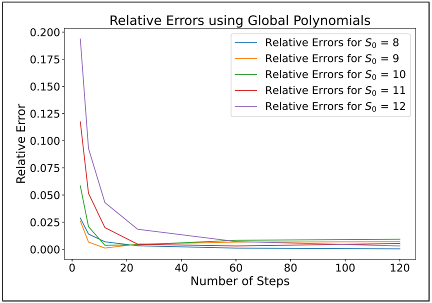


Figure 1: Relative Errors of American Put Options with parameters given in Table 1 and $\nu_0 = 0.0625$ calculated under the Heston Model.

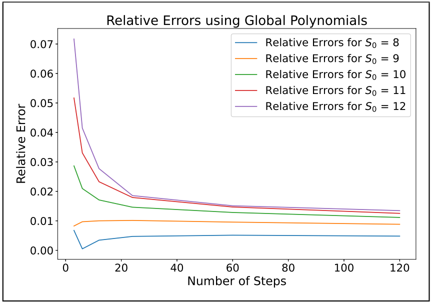


Figure 2: Relative Errors of American Put Option values with parameters given in Table 1 and $\nu_0 = 0.25$ calculated under the Heston Model.

# American and Bermudan Option Pricing when Feller condition does not hold

- As the variance must be positive, the Feller condition must be satisfied. In this case, Feller condition is given by

$$2\kappa\bar{\nu} > \gamma^2.$$

The Feller condition guarantees that the process $\nu_t$ cannot reach zero;
- The reference prices used in the following examples are taken from Fang and Oosterless[10];
- Parameters for both American and Bermudan option pricing are given in the table below.

| $K$ | $v_0$ | $T$ | $r$ | $\gamma$ | $\bar{\nu}$ | $\kappa$ | $\rho$ |
|---|---|---|---|---|---|---|---|
| 100 | 0.0348 | 0.25 | 0.04 | 0.39 | 0.0348 | 1.15 | $-0.64$ |

Table 2: Parameters for pricing American and Bermuda options under the Heston Model where Feller condition does not hold.

# American options

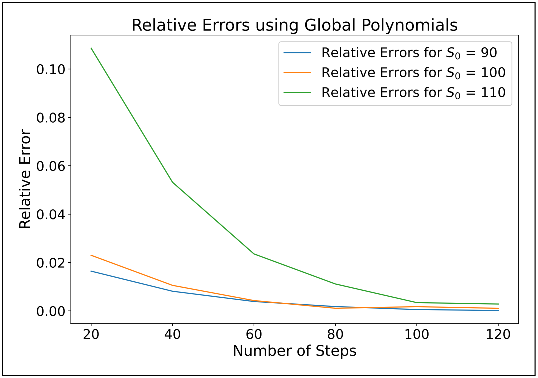


Figure 3: Relative Errors of American Put Option values with parameters given in Table 2 calculated under the Heston Model.

# Bermudan Options with 20 exercise dates

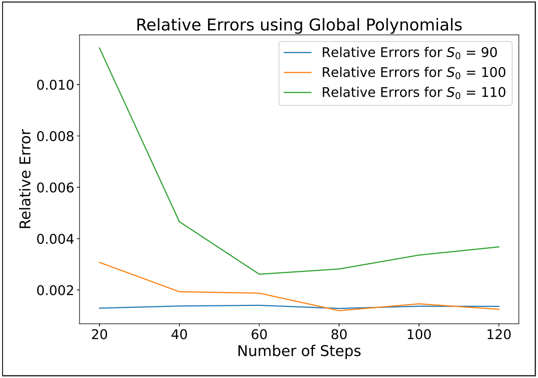


Figure 4: Relative Errors of Bermudan Option values with 20 exercise dates parameters given in Table 2 calculated under the Heston Model.

# Bermudan Options with 40 exercise dates

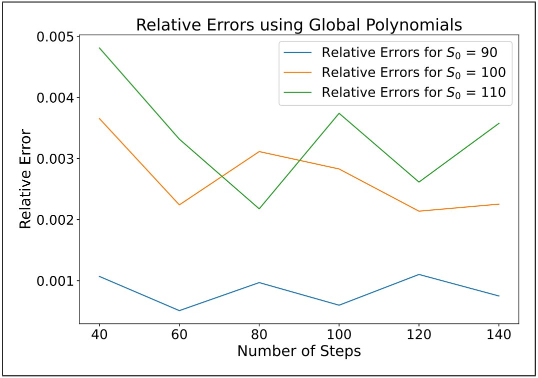


Figure 5: Relative Errors of Bermudan Option values with 40 exercise dates and parameters given in Table 2 calculated under the Heston Model.

# Bermudan Options with 60 exercise dates

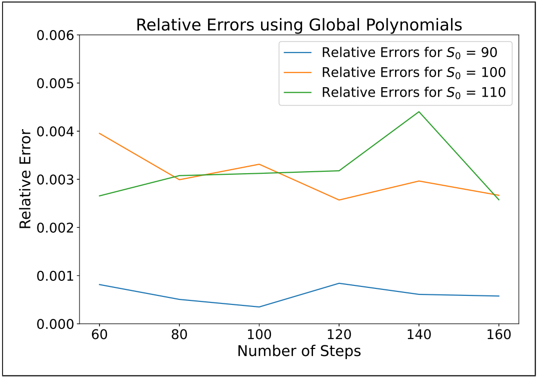


Figure 6: Relative Errors of Bermudan Option values with 60 exercise dates and parameters given in Table 2 calculated under the Heston Model.

# AES of the Double Heston Model

- The same setting as for the Heston Model;
- Parameters and reference prices used in the following examples are taken from Zhang and Feng[12].

# American Options: $T = 0.25$ and $T = 0.5$

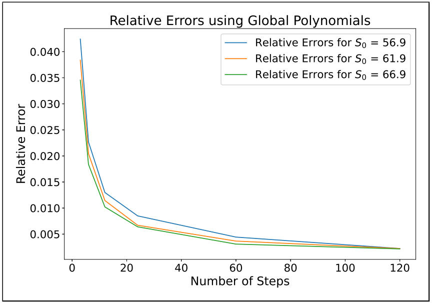


Figure 7: Relative Errors of American Put Option values when $T = 0.25$ calculated under the Double Heston Model.

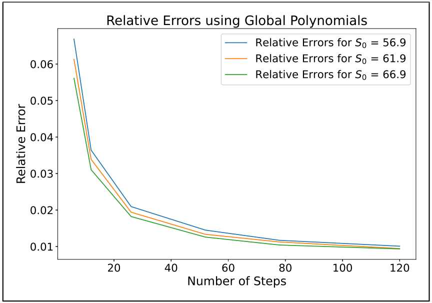


Figure 8: Relative Errors of American Put Option values when $T = 0.5$ calculated under the Double Heston Model.

# American Options: $T = 1$

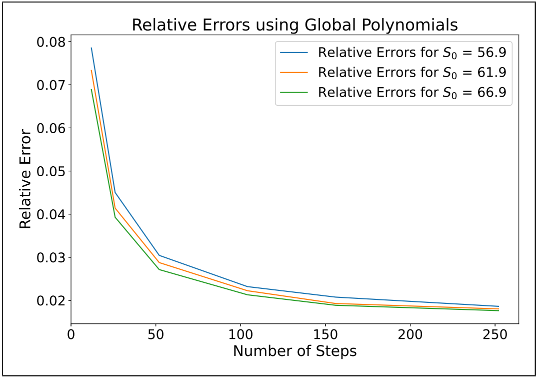


Figure 9: Relative Errors of American Put Option values when $T = 1$ calculated under the Double Heston Model.

# Euler vs AES

Euler vs AES when pricing Bermudan Options when the number of exercise dates (20) equals the number of steps (20) for different initial asset prices.

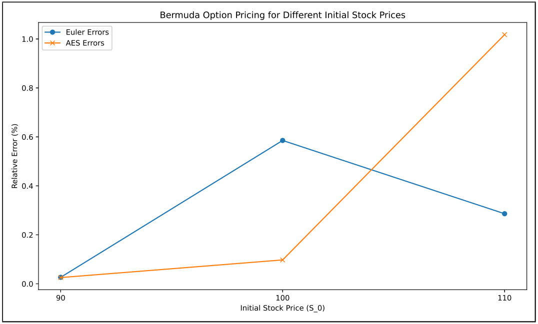


Figure 10: Relative Errors of Bermudan Options with 20 exercise dates and parameters given in Table 2 calculated using AES and Euler.

# Euler vs AES

Euler vs AES when pricing biweekly Bermudan Options when the number of exercise dates equals the number of steps for different time to maturities.

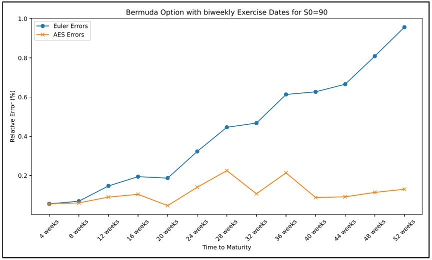


Figure 11: Relative Errors of biweekly Bermudan Option values with parameters given in Table 2 calculated under the Heston Model.

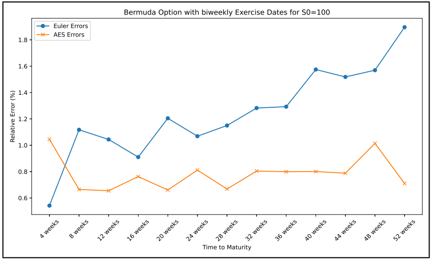


Figure 12: Relative Errors of biweekly Bermudan Option values with parameters given in Table 2 calculated using AES and Euler.

# Euler vs AES

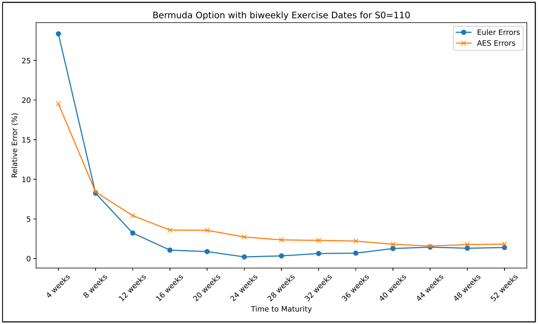


Figure 13: Relative Errors of biweekly Bermudan Option values with parameters given in Table 2 calculated under the Heston Model.

# Conclusion

- Extended the AES of the Heston Model to the Double Heston Model, and Brennan and Schwartz SDE;
- Pricing in-the-money and at-the-money American and Bermudan options does not require many steps;
- For Bermuda options, having as many steps as exercise dates is enough.

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Thank you for your attention!