Numerical Reconstruction of Volatility and Drift Rate from Market Observation Data

Miglena N. Koleva, Lubin G. Vulkov

Ruse University

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Introduction

- I. Bouchouev, V.Isakov, N. Valdivia, Recovery of volatility coefficient by linearization, Quant. Finance, 2002, 2, 257-263.

- Z.C. Deng, X.Y. Zhao, L.Yang, An inverse problem for reconstruction option **drift** from market observation data, Boundary Value Problems, vol.37, 2021

Formulation of the inverse problems

We consider the opportunities to arbitrage in the financial market. Thus, the following backward parabolic equation is derived

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2(t,S)S^2\frac{\partial^2 u}{\partial S^2} + \mu(S)S\frac{\partial u}{\partial S} - ru = 0, \tag{1}$$

The final time condition at the maturity for a binary option is specified by

$$u(T,S) = H(S-K) = \begin{cases} 1, & S \geq K \\ 0, & S < K. \end{cases} \tag{2}$$

It is natural to desire the drift function μ from the observed market price of options for different K and/or T and current time t^* with stock price S^* . In the continuous-time setting, this amounts to the following inverse problem.

Problem 1. Determine approximately the volatility σ and drift μ, such that the solution of (1), (2) fits the current market prices of options at (t^*, S^*) for *different strikes* K

$$u(t^*, S^*; K, T) = u^*(K, T) \tag{3}$$

and fixed maturity T.
We will solve this problem in the case of $\sigma^2(S)$ by Dupire technique that the price $u(T, K)$ of the binary option satisfies the adjoint equation

$$\frac{\partial u}{\partial T} - \frac{1}{2}\sigma^2(K)K^2\frac{\partial^2 u}{\partial K^2} + \mu(K)\frac{\partial u}{\partial K} + ru = 0, \quad (K, T) \in (0, \infty) \times (0, t), \tag{4}$$

$$u(t, S; T, K)|_{T=t} = H(S - K), \quad K \in (0, \infty), \tag{5}$$

The change of the variables

$$v(t, x) = u(T, K), \quad x = \ln\frac{K}{S^*}, \quad K = S^*e^x, \quad \tau = T - t, \tag{6}$$

$$a(x) = \mu(K) = \mu(S^*e^x), \quad \sigma^2 := \sigma^2(K) = \sigma^2(S^*e^x), \tag{7}$$

transforms the problem (4) to the following one

Problem 2. Cauchy problem of the parabolic equation

$$\frac{\partial v}{\partial \tau} - \frac{1}{2}\sigma^2(x)\frac{\partial^2 v}{\partial x^2} + \left(\frac{1}{2}\sigma^2(x) + a(x)\right)\frac{\partial v}{\partial x} + rv = 0, \quad (8)$$

$$v(0, x) = H(-x), \;\; x \in \mathbb{R}, \;\; \tau \in (0, \tau^*), \;\; \tau^* = T - t^* \quad (9)$$

where we have assume that the volatility doesn't depend on time. The boundary conditions we take as follows

$$v(-L, \tau) = 1, \;\; v(L, \tau) = 0. \quad (10)$$

The **extra** condition (3) is transformed into

$$v(\tau^*, x) = v^*(x) = u^*(S^* e^x, \tau + r), \;\; x \in \mathbb{R}. \quad (11)$$

The problem (8)-(11) is an inverse nonlinear problem with the unknown functions $\sigma^2(x)$, $a(x)$.

The linearization of volatility method

Following the idea of the papers [I. Bouchouev, V.Isakov, N. Valdivia, 2002] and due to mean reversion volatility, it is natural to assume that

$$\frac{1}{2}\sigma^2(x) = \frac{1}{2}\sigma_0^2 + f(x), \tag{12}$$

where f is a small perturbation continuous function of the constant σ_0^2. To derive the linearized inverse problem, let

$$v = V_0 + V + \widehat{v}. \tag{13}$$

The function V_0 solves (8)-(11) with known σ_0^2 and $\widehat{v}$ is *quadratic* always small with respect to f, while the principal linear term V satisfies the following problem for $\tau \in (0, \tau^*)$:

$$\frac{\partial V}{\partial t} - \frac{1}{2}\sigma_0^2 \frac{\partial^2 V}{\partial x^2} + \left(\frac{1}{2}\sigma_0^2 + a(x)\right)\frac{\partial V}{\partial x} + rV = f(x)\left(\frac{\partial^2 V_0}{\partial x^2} - \frac{\partial V_0}{\partial x}\right), \tag{14}$$

$$V(0,x) = V^0(x) = H_1(x) - H_2(x); \quad V(-L,\tau) = V(L,\tau) = 0, \tag{15}$$

$$V(\tau^*, x) = V^*(x). \tag{16}$$

Here it is assumed that $a(x)$ is already known function, determined after solving the inverse problem :

$$\frac{\partial V_0}{\partial \tau} - \frac{1}{2}\sigma_0^2 \frac{\partial^2 V_0}{\partial x^2} + \left(\frac{1}{2}\sigma_0^2 + a(x)\right)\frac{\partial V_0}{\partial x} + rV_0 = 0, \tag{17}$$

$$V_0(0,x) = V_0^0(x) = H_2(x), \quad V_0(-L,\tau) = 1, \quad V_0(L,\tau) = 0, \tag{18}$$

$$V_0(\tau^*, x) = V_0^*(x), \quad x \in \mathbb{R}, \tag{19}$$

where $H_1(x)$, $H_2(x)$ are two smoothed approximations of $H(-x)$

$$H(-x) = V_0(0,x) + V(0,x) + \widehat{v}_0 = H_1(x) + \widehat{v}_0 \text{ and } V_0^*(x) = v^*(x) - V^*(x). \tag{20}$$

Identification of $a(x)$

Now, we present the method for identification $a(x)$. Let $w_0(\tau, x) = \frac{\partial V_0}{\partial \tau}(\tau, x)$ for $(\tau, x) \in \overline{Q}_{\tau^*} = [0, \tau^*] \times [-L, L]$. We differentiate equation (17) with respect to τ to obtain

$$\frac{\partial w_0}{\partial \tau} - \frac{1}{2}\sigma_0^2 \frac{\partial^2 w_0}{\partial x^2} + \left(\frac{1}{2}\sigma_0^2 + a(x)\right) \frac{\partial w_0}{\partial x} + r w_0 = 0, \quad (\tau, x) \in Q_{\tau^*}$$

By the condition (19) and equation (17) at $\tau = \tau^*$, we find

$$a(x) = \frac{-w_0(\tau^*, x) + \frac{1}{2}\sigma_0^2 V_{0xx}^*(x) + r V_0^*(x)}{V_{0x}^*(x)} - \frac{1}{2}\sigma_0^2,$$

where

$$V_{0x}^* = \frac{\partial V_0^*}{\partial x}, \quad V_{0xx}^* = \frac{\partial^2 V_0^*}{\partial^2 x}.$$

Thus, the problem (17)-(19) is now equivalent to the following one: the PDE

$$\frac{\partial w_0}{\partial \tau}-\frac{1}{2}\sigma_0^2\frac{\partial^2 w_0}{\partial x^2}+\frac{-w_0(\tau^*,x)+\frac{1}{2}\sigma_0^2 V^*_{0xx}(x)+rV^*_0(x)}{V^*_{0x}(x)}\frac{\partial w_0}{\partial x}+rw_0=0,\quad (\tau,x) \tag{21}$$

and initial condition $w_0^0(x)$, obtained from (17) at initial time

$$w_0^0(x)=\frac{1}{2}\sigma_0^2 V^0_{0xx}(x)-\left(-w(\tau^*,x)+\frac{1}{2}\sigma_0^2 V^*_{0xx}+rV^*_0\right)\frac{V^0_{0x}(x)}{V^*_{0x}(x)}-rV^0_0, \tag{22}$$

and boundary conditions

$$w_0(\tau,-L)=0,\quad w_0(\tau,L)=0. \tag{23}$$

Identification of $f(x)$

Let $w(\tau, x) = \frac{\partial V}{\partial \tau}(\tau, x)$ for $(\tau, x) \in \overline{Q}_{\tau^*}$. Differentiating equation (14) with respect to τ yields

$$\frac{\partial w}{\partial t} - \frac{1}{2}\sigma_0^2 \frac{\partial^2 w}{\partial x^2} + \left(\frac{1}{2}\sigma_0^2 + a(x)\right)\frac{\partial w}{\partial x} + rw = f(x) f_{0\tau}, \quad (\tau, x) \in Q_{\tau^*},$$

where

$$f_0(\tau, x) = \frac{\partial^2 V_0}{\partial x^2}(\tau, x) - \frac{\partial V_0}{\partial x}(\tau, x), \quad f_{0\tau} = \frac{\partial f_0}{\partial \tau} = \frac{\partial^2 w_0}{\partial x^2}(\tau, x) - \frac{\partial w_0}{\partial x}(\tau, x).$$

By the equation (17), observation condition (19) and $\tau = \tau^*$, we find

$$f(x) = \frac{w(\tau^*, x) - \frac{1}{2}\sigma_0^2 V^*_{xx}(x) + \left(\frac{1}{2}\sigma_0^2 + a(x)\right) V^*_x(x) + rV^*(x)}{f_0(\tau^*, x)}.$$

Thus, the problem (17)-(19) is now equivalent to the following one: the PDE

$$\frac{\partial w}{\partial t} - \frac{1}{2}\sigma_0^2 \frac{\partial^2 w}{\partial x^2} + \left(\frac{1}{2}\sigma_0^2 + a(x)\right)\frac{\partial w}{\partial x} + rw \tag{24}$$

$$= \left(\frac{w(\tau^*, x) - \frac{1}{2}\sigma_0^2 V_{xx}^*(x) + \left(\frac{1}{2}\sigma_0^2 + a(x)\right) V_x^*(x) + rV^*(x)}{f_0(\tau^*, x)}\right) f_{0\tau}(x), \tag{25}$$

with initial condition

$$w^0(x) = \frac{1}{2}\sigma_0^2 \frac{\partial^2 V^0(x)}{\partial x^2} - \left(\frac{1}{2}\sigma_0^2 + a(x)\right)\frac{\partial V^0(x)}{\partial x} - r\partial V^0(x) + f(0) f_0(0, x) \tag{26}$$

and boundary conditions

$$w(\tau, -L) = w(\tau, L) = 0. \tag{27}$$

One possible choice for the measurements and initial conditions is:

$$V_0^0(x) = H(-x) \approx H_2(x), \quad V^0(x) = 0 \;\text{ i.e. }\; H_1(x) = H_2(x),$$
$$V^*(x) = v^*(x), \quad V_0^*(x) = 0.$$

Note that for this choice the initial condition (26) simplifies

$$w^0(x) = f(0) f_0(0, x). \tag{28}$$

Numerical Method

Define uniform spatial and temporal meshes:

$$x_i = -L + i\triangle x, \quad i = 0, 2, \ldots, N, \quad \triangle x = 2L/N,$$

$$\tau_n = n\triangle\tau, \quad n = 0, 1, \ldots, M, \quad \triangle\tau = T_1/M.$$

Denote by v_i^n the numerical solution v at grid node (x_i, t_n) and

$$v_{\mathring{x},i}{}^n = \frac{v_{i+1}^n - v_{i-1}^n}{2h},$$

$$v_{\bar{x}x,i}{}^n = \frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{h^2}.$$

Iteration processes with respect to $a(x)$, $f(x)$

Algorithm

1) Set $v^*(x)$, $V_0^*(x)$, tol, $w^M = [w^M]^{(0)} = 0$, $w_0^M = [w_0^M]^{(0)} = 0$, $k_a = 0$, $k_f = 0$, model and mesh parameters;

Recovering $a(x)$:

2) $k_a := k_a + 1$. Find $a_i^{(k_a)}$, $i = 1, 2, \dots, N-1$ from

$$a_i^{(k_a)} = \frac{-(w_0)_i^M + \frac{1}{2}\sigma_0^2 (V_0)^*_{\bar{x}x,i} + r(V_0)^*_i}{(V_0)^*_{\mathring{x},i}} - \frac{1}{2}\sigma_0^2;$$

3) Solve the discretization of (21)-(23) to find $[(w_0)_i^n]^{(k_a)} = (w_0)_i^n$, $i = 0, 1, \dots, N$, $n = 0, 1, \dots, M$;

4) If $\| [(w_0)_i^M]^{(k_a)} - [(w_0)_i^M]^{(k_a-1)} \| < tol$, stop the iteration procedure, $a_i := a_i^{(k_a)}$; otherwise, go to step 2) to update $(w_0)_i^M$ and $a_i^{(k_a)}$;

Recovering $f(x)$:

5) $k_f := k_f + 1$. Find $f_i^{(k_f)}$, $i = 1, 2, \dots, N-1$ from

$$f_i^{(k_f)} = \frac{w_i^M - \frac{1}{2}\sigma_0^2 V^*_{\bar{x}x,i} + \left(\frac{1}{2}\sigma_0^2 + a(x)\right) V^*_{\mathring{x},i} + rV_i^*}{(f_0)_i^M};$$

6) Solve the discretization of (24)-(27) to find $[w_i^n]^{(k_f)} = w_i^n$, $i = 0, 1, \dots, N$, $n = 0, 1, \dots, M$;

7) If $\| [w_i^M]^{(k_f)} - [w_i^M]^{(k_f-1)} \| < tol$, stop the iteration procedure, $f_i := f_i^{(k_f)}$; otherwise, go to step 5) to update w_i^M and $f_i^{(k_f)}$;

8) Solve the discretized direct problem (8)-(11) for the recovered a_i and f_i, $i = 1, 2, \dots, N-1$.

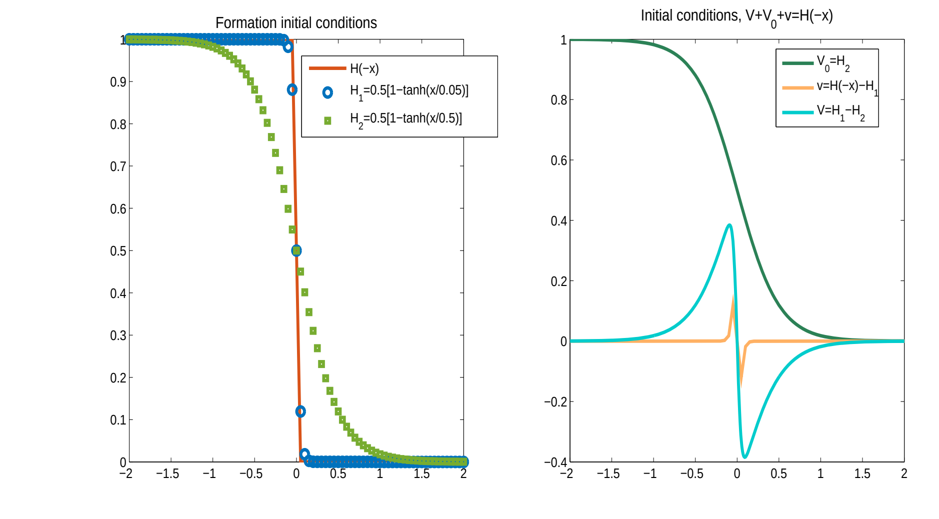
Numerical Simulations

Computational details:

- $[-2, 2]$, $\tau^* = 1$;
- $N = 80$, $\triangle\tau = h^2$, $tol = 1.e - 5$;
- Measurements are obtained from the numerical solution of the direct problem with exact $a(x)$ and $f(x)$ (*syntectic data*);
- $\mathcal{E}_\infty(N) = \max\limits_{0 \le i \le N} |v(x_i, T) - \overline{v}_i^M|$, $\mathcal{E}_2(N) = \sqrt{\sum\limits_{i=0}^{N} h(v(x_i, T) - \overline{v}_i^M)^2}$

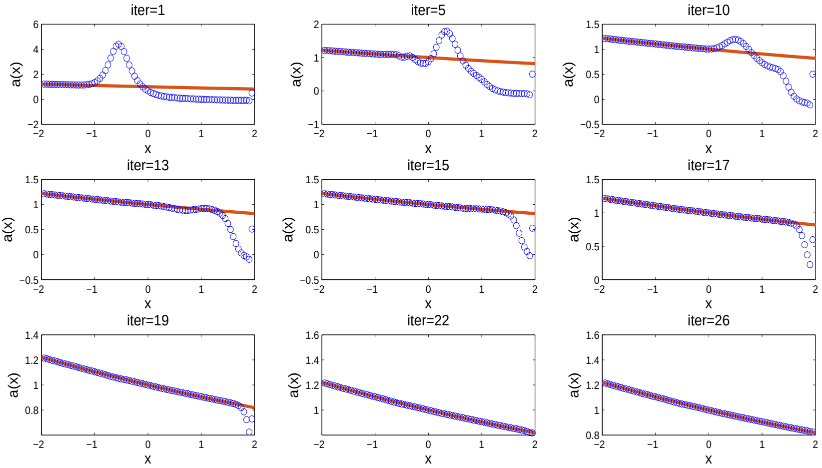
  -$v(x_i, T)$ is the numerical solution of (1)-(2) at final time with **exact** $a(x)$ and $f(x)$,
  - $\overline{v}_i^M$ is the numerical solution of (1)-(2) at final time with **recovered** $a(x)$ and $f(x)$;
- *convergence rate:* $CR_\infty = \log_2 \frac{\mathcal{E}_\infty(2N)}{\mathcal{E}_\infty(N)}$, $CR_2 = \log_2 \frac{\mathcal{E}_2(2N)}{\mathcal{E}_2(N)}$.

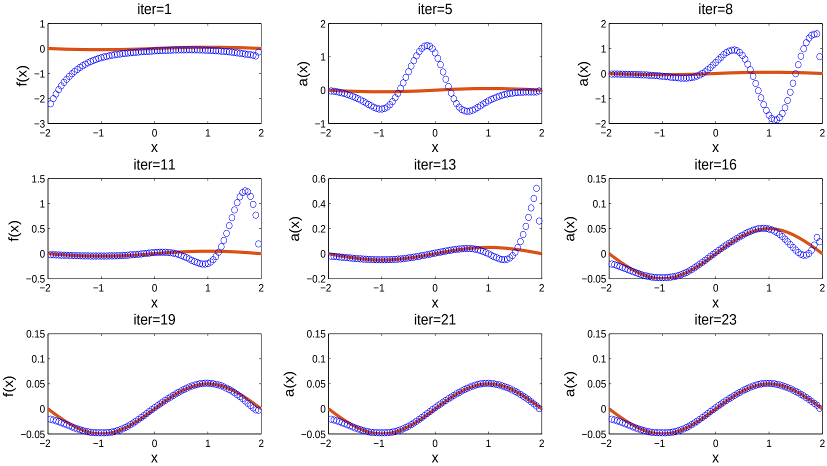
Initial conditions



Test problem 1

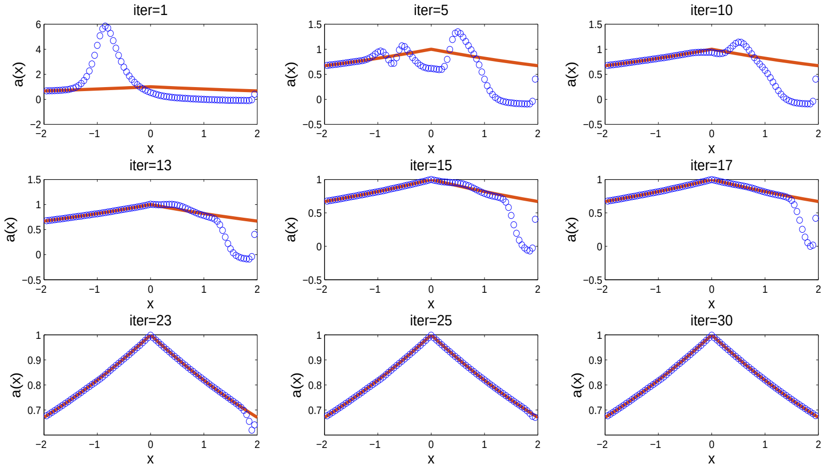
$f(x) = \frac{1}{2}\sin\frac{\pi x}{2}$, $\sigma_0 = 0.2$, $a(x) = e^{-x/10}$ [Deng, Zhao, Yang, 2021]

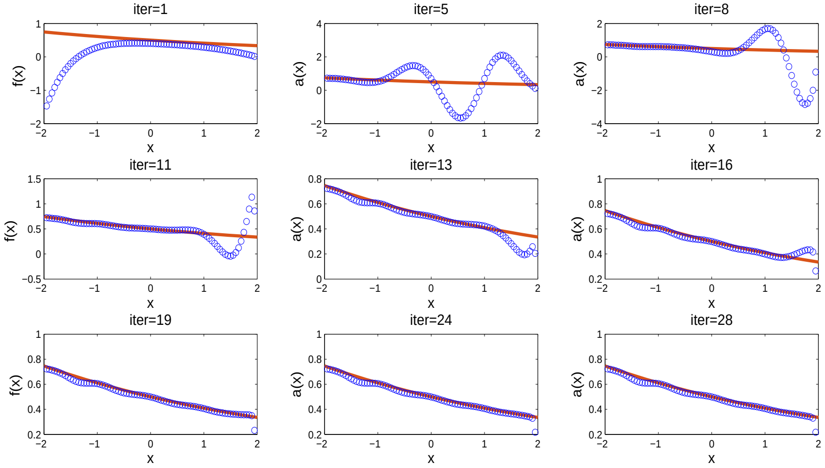



iter=1
iter=5
iter=8
iter=11
iter=13
iter=16
iter=19
iter=21
iter=23
f(x)
a(x)
x

Test problem 2

$f(x) = \frac{1}{2}e^{-0.2x}$, $\sigma_0 = 0.2$, $a(x) = (e^{|x|})^{-1/5}$ [Deng, Zhao, Yang, 2021]




iter=1
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iter=16
iter=19
iter=24
iter=28
f(x)
a(x)
x

Table: Errors and spatial order of convergence

N	$\mathcal{E}_\infty(N)$	CR_∞	$\mathcal{E}_\infty(N)$	CR_2
40	3.6746e-3		3.5664e-3	
80	1.5574e-3	1.2394	1.0292e-3	1.7929
160	3.9749e-4	1.9701	2.5912e-4	1.9899

Conclusions

- Numerically solving inverse coefficient and source problem for recovering volatility and drift rate of the binary call option in Black-Scholes model;
- We formulate two inverse problems, which are transformed to forward problems with non-local terms in the differential operator and the initial condition.
- We construct iterative numerical algorithm for solving these problems;
- The unknown volatility and drift rate are determined with optimal accuracy for moderate number of iterations;
- The spatial order of convergence of the recovered solution is second.