Pricing and replication of the prepayment option introducing a non-hedgeable risk factor

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Fixed rate mortgages and swaps

- Fixed rate mortgages expose financial institutions to interest rate risk.
- The combination of fixed rate mortgages and suitable floating rate notes (FRNs) is equivalent to (amortizing) interest rate swaps (IRSs).
- The interest rate risk is fully hedged using (amortizing) swaps.

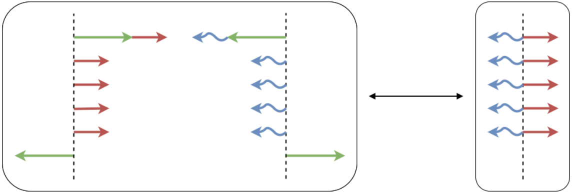


Figure 1: Equivalence between an IRS (right) and the sum of a fixed rate interest-only mortgage and an FRN (left). Green: notional payments. Red: fixed rate payments. Blue: floating rate payments.

Embedded prepayment option

- Most mortgage contracts allow the owner to (partially) repay in advance the mortgage notional, namely to *prepay*.
- The *prepayment right* is an *embedded prepayment option* (EPO).
- Prepayment leads to a mismatch between the cash flows of the original hedge and of the mortgage $\longrightarrow$ *interest rate risk!*

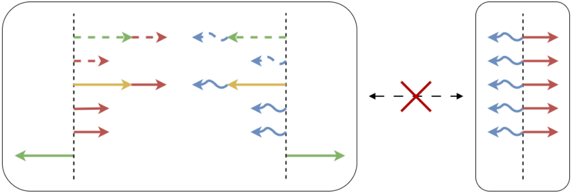


Figure 2: Effect of prepayment on future cash flows. Green: notional payments. Red: fixed rate payments. Blue: floating rate payments. Yellow: notional prepayments. Dotted: unrealized future cash flows.

Stochastic notional profile

- Portfolio of mortgages: fixed rate K, contractual notional profile $N_c(t)$ (with $N_0 = N_c(t_0)$), and payment dates $\mathrm{T}_c = \{t_1, \ldots, t_n\}$.
- *Annualized unconditional instantaneous* prepayment at t, $\Lambda(t)$.
- The realized notional, $N_r(t)$, and the EPO notional, $N(t)$ are:

$$N_r(t) = \left[N_c(t) - N_0 \int_{t_0}^{t} \Lambda(\tau)\mathrm{d}\tau\right]^+, \quad N(t) = N_c(t) - N_r(t).$$

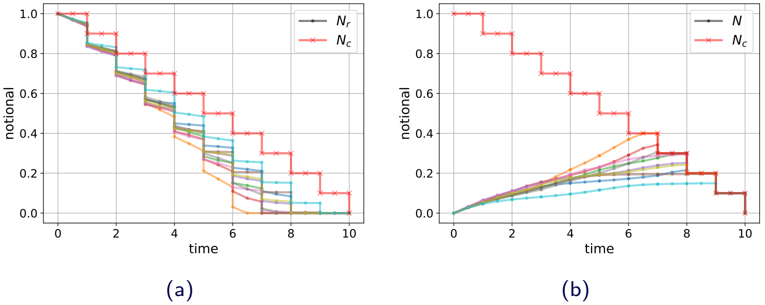


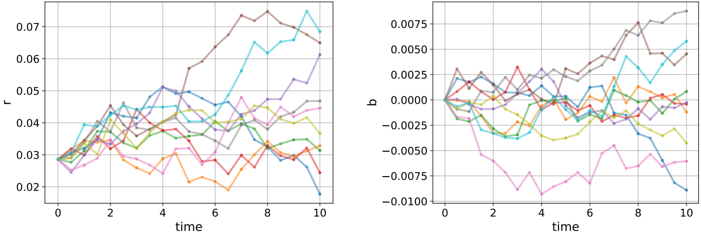
Figure 3: Notional profiles. Left: mortgage realized notional. Right: EPO notional.

Model dynamics

- On the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, we consider the two risk factors $r(t)$ and $b(t)$, $t \in [t_0, t_n]$, with dynamics:

$$\mathrm{d}r(t) = \alpha_r^{\mathbb{P}}(\theta_r^{\mathbb{P}}(t) - r(t))\mathrm{d}t + \eta_r \mathrm{d}W_r^{\mathbb{P}}(t), \quad r(t_0) = r_0 \in \mathbb{R},$$
$$\mathrm{d}b(t) = \alpha_b^{\mathbb{P}}(\theta_b^{\mathbb{P}} - b(t))\mathrm{d}t + \eta_b \mathrm{d}W_b^{\mathbb{P}}(t), \quad b(t_0) = b_0 \in \mathbb{R},$$

  where $\mathrm{d}W_r^{\mathbb{P}}(t)\mathrm{d}W_b^{\mathbb{P}}(t) = \rho \mathrm{d}t$, $\rho \in [-1, 1]$.
- $r(t)$ is the short rate, while $b(t)$ is a *non-hedgeable* risk factor.



(a) (b)

Figure 4: Left: hedgeable risk factor $r(t)$. Right: non-hedgeable risk factor $b(t)$.

Rate incentive function

- We define the *rate incentive* as:

$$\varepsilon(t) = K - \kappa(t) = g(t, r(t)),$$

with κ a reference rate and g a suitable function.

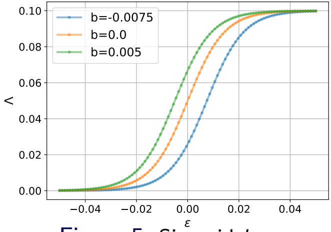


Figure 5: Sigmoid h.

- The *annualized unconditional instantaneous prepayment* $\Lambda(t)$ is modelled using a sigmoid h:

$$\Lambda(t) = h(t, \varepsilon(t), b(t)),$$
$$h(t, \epsilon, b) = l + \frac{u - l}{2}\Big[\tanh\big(a(\epsilon + b)\big) + 1\Big], \quad l, u, a \in \mathbb{R}.$$

EPO cash flows and value

- The EPO has multiple *cash flows*, occurring at $t_j \in \mathrm{T}_c$, given by:

$$H(t_j) = \left[\int_{t_{j-1}}^{t_j} N(t)\mathrm{d}t \right] \Big(K - L(t_{j-1}; t_{j-1}, t_j) \Big),$$

for the reference floating rate $L(t_{j-1}; t_{j-1}, t_j)$ fixed at time t_{j-1}, and:

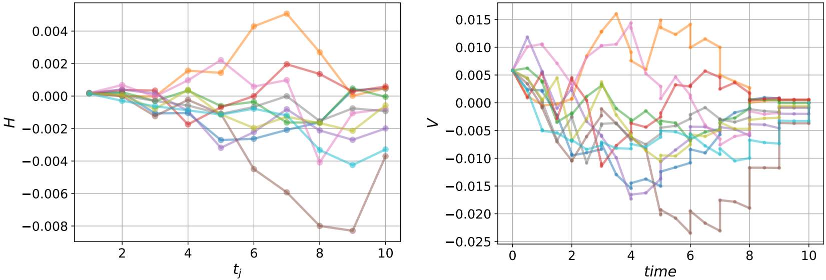
$$N(t) = N_c(t) - N_r(t) = \min \left(N_c(t), N_0 \int_{t_0}^{t} \Lambda(\tau)\mathrm{d}\tau \right).$$

- The EPO *value process* reads:

$$V(t, r, b, N) = \mathbb{E}^{\mathbb{Q}} \left[\sum_{t_j > t} \frac{M(t)}{M(t_j)} H(t_j) \middle| \mathcal{F}(t) \right],$$

where $\mathbb{Q}$ is a – *non unique* – *equivalent martingale measure* (EMM) associated with the money savings account $M(t)$ as numéraire.

Cash flows and value process



(a) (b)

Figure 6: Left: cash flows. Right: value process.

Multiple risk neutral measures

- The valuation task requires the knowledge of a *risk neutral dynamics*:

$$\mathrm{d}r(t) = (\cdots)\mathrm{d}t + \eta_r \mathrm{d}W_r^{\mathbb{Q}_\lambda}(t), \qquad r(t_0) = r_0 \in \mathbb{R},$$
$$\mathrm{d}b(t) = (\cdots)\mathrm{d}t + \eta_b \mathrm{d}W_b^{\mathbb{Q}_\lambda}(t), \qquad b(t_0) = b_0 \in \mathbb{R}.$$

- The *change of measure* is driven by the relationship:

$$\mathrm{d}W_r^{\mathbb{Q}_\lambda}(t) = \lambda_r(t)\mathrm{d}t + \mathrm{d}W_r^{\mathbb{P}}(t),$$
$$\mathrm{d}W_b^{\mathbb{Q}_\lambda}(t) = \lambda_b(t)\mathrm{d}t + \mathrm{d}W_b^{\mathbb{P}}(t).$$

- Since b is non hedgeable, the market price of risk λ_b is undetermined.
- The value process is a function of λ_b, namely $V(t, r, b, N; \lambda_b)$.

Robust (static) replication

- The wealth invested in the EPO and in the replication are:

$$W_{EPO}(t;\lambda_b) = V(t;\lambda_b) + C_{EPO}(t),$$
$$W_{REP}(t;\mathbf{w}) = \sum_i w_i V_i(t) + C_{REP}(t),$$

where w_i and V_i are notional and value of the i-th hedging instrument, and C_{EPO} and C_{REP} are suitable cash accounts.

- A loss function is defined as:

$$L(\mathbf{w};\lambda_b) = \int_{t_0}^{t_n} \mathbb{E}^{\mathbb{Q}_\lambda}\left[\Big(W_{EPO}(t;\lambda_b) - W_{REP}(t;\mathbf{w})\Big)^2\right]\mathrm{d}t.$$

- The optimal *robust replication* $\mathbf{w}^*$ is obtained – if it exists – solving:

$$\inf_{\mathbf{w}} \sup_{\lambda_b} L(\mathbf{w};\lambda_b).$$

"Affine" change of measure

- By restricting λ_b to *affine processes* $\lambda_b(t) = \lambda_0 + \lambda_1 b(t)$, with $(\lambda_0, \lambda_1) \in \mathbb{R}^2$, the risk neutral dynamics for b reads:

$$\mathrm{d}b(t) = \alpha_b^{\mathbb{Q}_\lambda}(\theta_b^{\mathbb{Q}_\lambda} - b(t))\mathrm{d}t + \eta_b \mathrm{d}W_b^{\mathbb{Q}_\lambda}(t),$$

  for $\alpha_b^{\mathbb{Q}_\lambda} = \alpha_b^{\mathbb{P}} + \eta_b\lambda_1$ and $\theta_b^{\mathbb{Q}_\lambda} = \frac{\alpha_b^{\mathbb{P}}\theta_b^{\mathbb{P}} - \eta_b\lambda_0}{\alpha_b^{\mathbb{Q}_\lambda}}$.

- We *bound the search domain* to:

$$D_\lambda = \left\{(\lambda_0, \lambda_1) : 0 < \alpha_b^{\mathbb{Q}_\lambda} \leq \overline{\alpha_b}, \quad |\theta_b^{\mathbb{Q}_\lambda}| \leq \overline{\theta_b}\right\}, \qquad \overline{\alpha_b}, \overline{\theta_b} \in \mathbb{R},$$

  with $\overline{\alpha_b}$ and $\overline{\theta_b}$ bounds depending on the risk attitude/belief of the seller.

Restricted domain

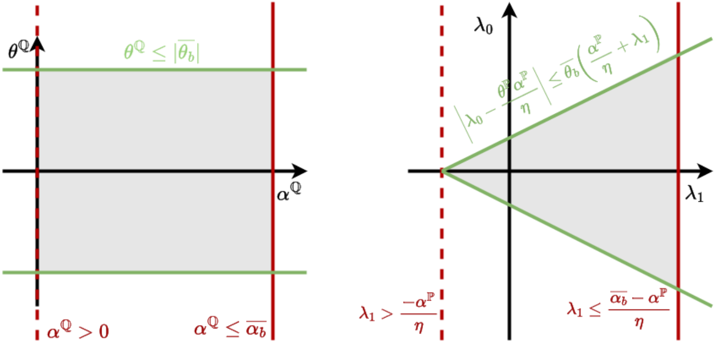


Figure 7: Restricted search domain D_λ.

"Optimal" solution and replication bounds

- We solve the restricted problem $\inf_{\mathbf{w}} \max_{\lambda_b \in D_\lambda} L(\mathbf{w}; \lambda_b)$ iteratively valuing the replication loss on a finite grid $D_{\lambda,0} \subset D_\lambda$.
- We observe a bang-bang behavior: the optimal replication under one measure is the worst against another one.

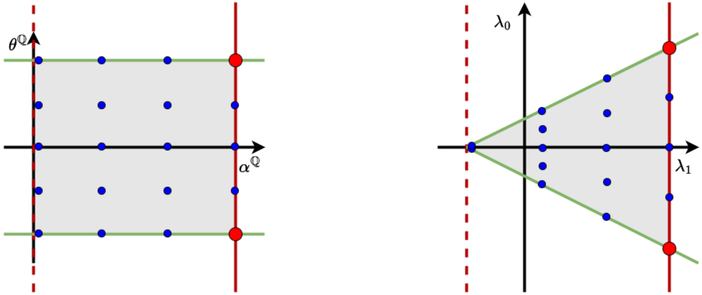
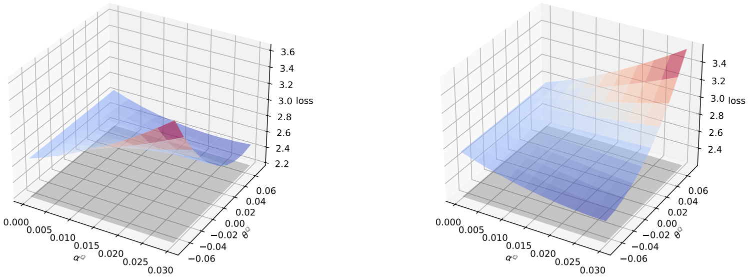


Figure 8: Discrete domain $D_{\lambda,0}$ and "worst" measures.

Loss – swap



(a)

(b)

Figure 9: Replication with single swap.

Loss – swaption

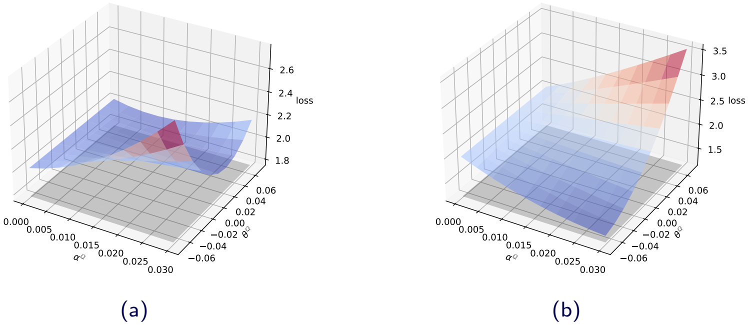
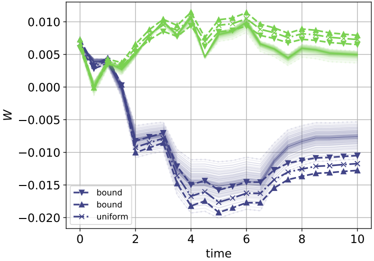
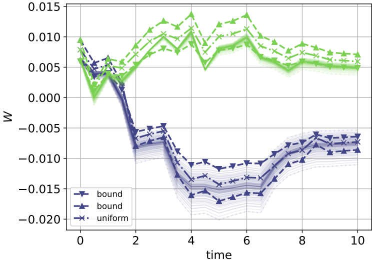


Figure 10: Replication with swap and swaption.

Replication paths



(a)



(b)

Figure 11: Left: single swap. Right: swap and swaption.

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