Beyond Affine Models:
On Inclusion of Random Parameters in Pricing Models

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Lech A. Grzelak
E-mail: L.A.Grzelak@uu.nl
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Advertisement, Must-See Presentations

- **Presentation** by Griselda Deelstra (ULB, Brussels, Belgium)
  Consistent Asset Modelling with Randomness in the Coefficients and Switches Between Regimes, Wolf et al. (2024).
- **Presentation** by Thomas van der Zwaard (Rabobank, UU, NL)
  Short-Rate Models With Smile and Applications to Valuation Adjustments, Zwaard et al. (2024).

Motivation

- Recent studies have shown that the standard models do not offer sufficient flexibility in pricing advanced derivatives, e.g., options on S&P and VIX.
- In Carr and Wu (2007), where the problem of insufficient skew was reported, and the remedy in terms of randomization was suggested: *"it would be tempting to try to capture stochastic skewness by randomizing the mean jump size parameter (...) However, randomizing either parameter is not amenable to analytic solution techniques that greatly aid econometric estimation."*
- The concept of randomizing is more fundamental, i.e., it represents the incorporation of the uncertainty of potentially hidden states that are not adequately captured by deterministic parameters.
- We can also consider randomization as a regime-switching method, with the states determined by the randomizing random variable.
- In this talk, we present the Randomized-Affine-Diffusion (RAnD) method Grzelak (2022a), which allows for efficient pricing of Affine Models with Random parameters.

Affine Models

- The stochastic model of interest can be expressed by the following stochastic differential form:

$$\mathrm{d}\mathbf{X}(t) = \boldsymbol{\mu}(t, \mathbf{X}(t))\mathrm{d}t + \boldsymbol{\sigma}(t, \mathbf{X}(t))\mathrm{d}\widetilde{\mathbf{W}}(t) + \mathbf{J}(t)^{\mathrm{T}}\mathrm{d}\mathbf{X}_{\mathcal{P}}(t),$$

  where $\widetilde{\mathbf{W}}(t)$ is a column vector of *independent* Brownian motions, $\boldsymbol{\mu}$, is a drift, $\boldsymbol{\sigma}$ corresponds to volatility, and $\mathbf{X}_{\mathcal{P}}(t)$ is a vector of orthogonal Poisson processes characterized by an intensity vector $\boldsymbol{\xi}$.
- We consider an orthogonal vector $\Theta = [\vartheta_1, \ldots, \vartheta_n]^{\mathrm{T}}$, $n \in \mathbb{N}$, where each ϑ_i is an independent, time-invariant, random variable [1].
- A realization of ϑ_i we indicate by θ_i, $\vartheta_i(\omega) = \theta_i$.
- We assume the model is affine for a realization of a random parameter.

[1] We consider here $n \in \mathbb{N}$ stochastic parameters, this is however not a necessary constraint.

Affine Models

- Affinity conditions require the following linearity of the model:

$$
\begin{aligned}
\boldsymbol{\mu}(t, \mathbf{X}(t)) &= a_0(\theta) + a_1(\theta)\mathbf{X}(t), \\
\boldsymbol{r}(t, \mathbf{X}(t)) &= r_0(\theta) + r_1(\theta)^{\mathsf{T}}\mathbf{X}(t), \\
(\boldsymbol{\sigma}(t, \mathbf{X}(t))\boldsymbol{\sigma}(t, \mathbf{X}(t))^{\mathsf{T}})_{i,j} &= (c_0(\theta))_{i,j} + (c_1(\theta))_{i,j}^{\mathsf{T}}\mathbf{X}_j(t), \\
\xi(t, \mathbf{X}(t)) &= l_0(\theta) + l_1(\theta)\mathbf{X}(t).
\end{aligned}
$$

- For a given realization of Θ, θ, we consider $\mathbf{X}_\theta(t) := \mathbf{X}(t)|\Theta = \theta$, $\mathbf{J}_\theta(t) := \mathbf{J}(t)|\Theta = \theta$, the discounted characteristic function is also of the following form (Duffie et al., 2000):

$$
\phi_{\mathbf{X}_\theta}(\mathbf{u}; t, T) = \mathbb{E}_t\left[\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s + i\mathbf{u}^{\mathsf{T}}\mathbf{X}_\theta(T)}\right] = \mathrm{e}^{A(\mathbf{u};\tau,\theta) + \mathbf{B}^{\mathsf{T}}(\mathbf{u};\tau,\theta)\mathbf{X}_\theta(t)},
$$

with the expectation under risk-neutral measure $\mathbb{Q}$ for $\tau = T - t$.

Affine Models and Randomization

- The coefficients $A := A(\mathbf{u};\tau,\theta)$ and $\mathbf{B} := \mathbf{B}^{\mathrm{T}}(\mathbf{u};\tau,\theta)$, satisfy complex-valued *Riccati* ODEs (Duffie et al., 2000):

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} = -r_0(\theta) + \mathbf{B}^{\mathrm{T}} a_0(\theta) + \frac{1}{2}\mathbf{B}^{\mathrm{T}} c_0(\theta)\mathbf{B} + l_0^{\mathrm{T}}\mathbb{E}\left[\mathrm{e}^{\mathbf{J}_\theta(\tau)\mathbf{B}} - 1\right],$$
$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\tau} = -r_1(\theta) + a_1(\theta)^{\mathrm{T}}\mathbf{B} + \frac{1}{2}\mathbf{B}^{\mathrm{T}} c_1(\theta)\mathbf{B} + l_1(\theta)^{\mathrm{T}}\mathbb{E}\left[\mathrm{e}^{\mathbf{J}_\theta(\tau)\mathbf{B}} - 1\right],$$

  where the expectation, $\mathbb{E}[\cdot]$, is taken with respect to the jump amplitude $\mathbf{J}_\theta(t)$.
- Then, for stochastic parameter ϑ, the ChF is given by:

$$\phi_{\mathbf{X}}(\mathbf{u};t,T) := \mathbb{E}_t\left[\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s + i\mathbf{u}^{\mathrm{T}}\mathbf{X}(T)}\right] = \mathbb{E}_t\left[\mathbb{E}_t\left[\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s + i\mathbf{u}^{\mathrm{T}}\mathbf{X}_\theta(T)} \middle| \Theta = \theta\right]\right].$$

- The inner expectation can be recognized as the conditional ChF; thus, by definition of the ChF and integration over all the parameter space, we find,

$$\phi_{\mathbf{X}}(\mathbf{u};t,T) = \mathbb{E}_t\left[\phi_{\mathbf{X}|\Theta}(\mathbf{u};t,T)\right] = \int_{\mathbb{R}^n} \phi_{\mathbf{X}|\Theta=\theta}(\mathbf{u};t,T) f_\Theta(\theta)\mathrm{d}\theta.$$

- We aim to provide numerically efficient methods for the computation of randomized ChF.

Affine Models and Randomization- The RAnD Method

- To determine the ChF of an affine model with a *randomized parameter*, one needs to integrate the parameter's probability density function- computationally expensive!
- This can be avoided, i.e., the complicated integrand can be factored into a set of pairs $\{\omega_n, \theta_n\}_{n=1}^N$, $N \in \mathbb{N}$, with a nonnegative "weights" function, $\omega_n \geq 0$, such that $\sum_{n=1}^N \omega_n = 1$ and specific, collocation, points θ_n,

$$\phi_{\mathbf{X}}(\mathbf{u}; t, T) = \int_{\mathbb{R}^n} \phi_{\mathbf{X}|\Theta=\theta}(\mathbf{u}; t, T) f_\Theta(\theta) \mathrm{d}\theta = \sum_{n=1}^N \omega_n \phi_{\mathbf{X}|\vartheta=\theta_n}(u; t, T) + \underline{\epsilon_N}.$$

- Once the number of evaluations, N, is low, we can significantly reduce the computational cost. The key element here, however, is that the pairs, $\{\omega_n, \theta_n\}_{n=1}^N$, cannot be chosen arbitrarily but need to be computed based on the parameter's distribution, ϑ.
- We follow the approach presented in (Golub and Welsch, 1969) where ω_n are the quadrature weights determined based on the moments of the random parameter, ϑ.
- Mixture distribution models have been studied in Brigo and Mercurio (2002), where the sum of (log)normal PDFs was analyzed. The model, although very flexible, was limited by a large number of model parameters.

Randomized Affine Models- The RAnD Method

Theorem (ChF for Randomized Affine Jump Diffusion Processes Grzelak (2022a))

Consider a random variable ϑ, with its PDF, $f_\vartheta(x)$, CDF, $F_\vartheta(x)$ and a realization θ, $\vartheta(\omega) = \theta$ such that for some $N \in \mathbb{N}$ the moments are finite, $\mathbb{E}[\vartheta^{2N}] < \infty$. Assuming that the corresponding ChF, $\phi_{\mathbf{X}|\vartheta=\theta}(\cdot)$, is well defined and $2N$ times differentiable w.r.t. θ, the unconditional ChF for the randomized **X**, *exists and is given by:*

$$\phi_{\mathbf{X}}(u;t,T) = \sum_{n=1}^{N} \omega_n \phi_{\mathbf{X}|\vartheta=\theta_n}(u;t,T) + \underline{\epsilon_N} = \sum_{n=1}^{N} \omega_n \mathrm{e}^{A(u;\tau,\theta_n)+\mathbf{B}^{\mathrm{T}}(u;\tau,\theta_n)\mathbf{X}(t)} + \underline{\epsilon_N},$$

where

$$\underline{\epsilon_N} = \frac{1}{(2N)!}\frac{\partial^{2N}}{\partial \xi^{2N}}\phi_{\mathbf{X}|\vartheta=\xi}(u;t,T),$$

for $a < \xi < b$ and where the pairs $\{\omega_n, \theta_n\}_{n=1}^{N}$ are the Gauss-quadrature weights and the nodes based on the parameter distribution, $f_\vartheta(\cdot)$, determined by $\zeta(\vartheta) : \mathbb{R} \to \{\omega_n, \theta_n\}_{n=1}^{N}$.

- We report exponential convergence!

Affine Models and Randomization

- The ChF of the randomized AD model is a weighted sum of a set of conditional ChFs evaluated at certain realizations, θ_n, of the underlying random parameter ϑ.
- The theorem shows the exponential decay of the error in terms of N-suggesting high precision for low N.
- When analytical moments are available, the computation of the corresponding points only requires the computation of a Cholesky decomposition and certain eigenvalues; it is, therefore, computationally cheap.
- Variables under closed under linear transformations allow for the tabulation of the corresponding quadrature points!

Table: Selected distributions for the stochastic parameters.

name	raw moment	domain
$\vartheta \sim \mathcal{U}([\hat{a}, \hat{b}])$	$\mathbb{E}[\vartheta^n] = \frac{\hat{b}^{n+1} - \hat{a}^{n+1}}{(n+1)(\hat{b} - \hat{a})}$	$[\hat{a}, \hat{b}]$
$\vartheta \sim \exp(\hat{a})$	$\mathbb{E}[\vartheta^n] = \frac{n!}{\hat{a}^n}$	$\mathbb{R}^+$
$\vartheta \sim \mathcal{N}(0, 1)$	$\mathbb{E}[\vartheta^n] = (n-1)!!$ if n even; 0 otherwise	$\mathbb{R}$
$\vartheta \sim \Gamma(\hat{a}, \hat{b})$	$\mathbb{E}[\vartheta^n] = \hat{b}^n \Gamma(n + \hat{a}) / \Gamma(\hat{b})$	$\mathbb{R}^+$
$\vartheta \sim \chi^2(\hat{a}, \hat{b})$	$\mathbb{E}[\vartheta^n] = 2^{n-1}(n-1)!(\hat{a} + n\hat{b}) + \sum_{j=1}^{n-1} \frac{(n-1)!2^{j-1}}{(n-j)!}(\hat{a} + j\hat{b})\mathbb{E}[\vartheta^{n-j}]$	$\mathbb{R}^+ \cup \{0\}$

PDF of Randomized Models

- The application of Fourier inversion to the randomized ChF yields,

$$f_{\mathbf{X}}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-iux} \sum_{n=1}^{N} \omega_n \phi_{\mathbf{X}|\vartheta=\theta_n}(u;t,T)\mathrm{d}u = \sum_{n=1}^{N} \omega_n f_{\mathbf{X}|\vartheta=\theta_n}(x).$$

  Since $\omega_1 + \cdots + \omega_N = 1$, $\omega_n \geq 0$, for $n = 1, \ldots, N$, which implies the density of the affine, randomized, system of SDEs, $\mathbf{X}(t)$ can be expressed as a, possibly multi-modal, mixture distribution.

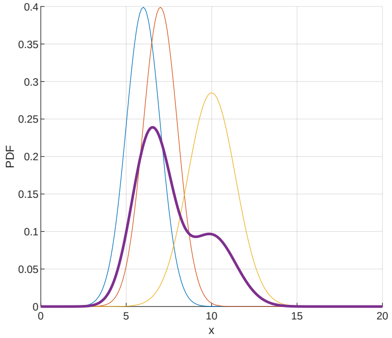

Figure: mixture PDF with three PDFs: $X \sim \mathcal{N}(6,1)$, $Y \sim \mathcal{N}(7,1)$, $Z \sim \mathcal{N}(10,1.4)$.

Pricing with Randomized Models

- The pricing will rely on a Fourier inversion method, namely the COS method (Fang and Oosterlee, 2008).
- The generic pricing equation is given by:

$$V(t_0) = \mathrm{e}^{-r(T-t_0)} {\sum_{k=0}^{N_c-1}}' \Re\left[\phi_{\mathbf{X}}\left(\frac{k\pi}{b-a}; t_0, T\right) \exp\left(-ik\pi\frac{a}{b-a}\right)\right] \cdot H_k + \epsilon_{c_1},$$

  where H_k for $k \geq 0$ are known in closed-form coefficients corresponding to the payoff function.
- We will use the H_k coefficients derived for European-style call/put options and options on VIX.
- Parameters, a and b are the *tuning* parameters used to determine the integration range Junike and Pankrashkin (2022); while the error ϵ_{c_1}, is exponentially decaying in N_c.

Randomized Black-Scholes Model

- We consider the randomized Black-Scholes, with σ random, and follow a uniform distribution, $\sigma \sim \mathcal{U}([\cdot,\cdot])$.
- The randomized Black-Scholes model follows the following SDE:

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t), \quad \sigma \sim \mathcal{U}([\hat{a},\hat{b}]), \quad \hat{a},\hat{b}, \in \mathbb{R}^+.$$

- The corresponding ChF for $X(t) = \log S(t)$, reads:

$$\phi_X(u;t_0,T) = \sum_{n=1}^{N} \omega_n \exp\left(iuX(t_0) + \left(r - \frac{1}{2}{\sigma_n}^2\right)iu(T-t_0) - \frac{1}{2}{\sigma_n}^2u^2(T-t_0)\right) + \epsilon_N.$$

- In the experiment we take: $\sigma \sim \mathcal{U}([0.1, 0.45])$.
- We note that $\{\omega_i,\theta_i\}_{i=1}^N$ can be computed for $\mathcal{U}([0,1])$ and then scaled appropriately.
- The Black-Scholes model with discrete σ_n realizations leads to the parametric local-volatility type of model Brigo and Mercurio (2002).

Implied Volatility Surface for the Randomized BS Model

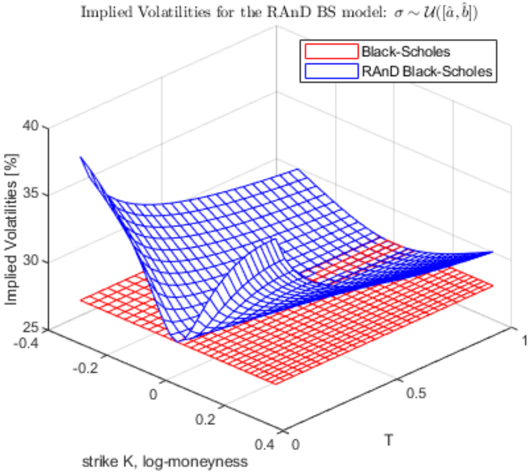

Figure: Right: Implied volatility surface for the RAnD BS model for $\sigma \sim \mathcal{U}([\hat{a}, \hat{b}])$.

The RAnD Bates Model

- The Bates model Bates (1996), under the $\mathbb{Q}$ measure, is described by the following system of SDEs:

$$\mathrm{d}S(t)/S(t) = \left(r - \lambda\mathbb{E}\left[\mathrm{e}^{J} - 1\right]\right)\mathrm{d}t + \sqrt{v(t)}\mathrm{d}W_x(t) + \left(\mathrm{e}^{J} - 1\right)\mathrm{d}X_{\mathcal{P}}(t),$$
$$\mathrm{d}v(t) = \kappa\left(\bar{v} - v(t)\right)\mathrm{d}t + \gamma\sqrt{v(t)}\mathrm{d}W_v(t),$$

  with Poisson process $X_{\mathcal{P}}(t)$, intensity λ, and normally distributed jump sizes, $J \sim \mathcal{N}(\mu_j, \sigma_j^2)$, with $\mathbb{E}[\mathrm{e}^{J}] = \mathrm{e}^{\mu_J + \frac{1}{2}\sigma_J^2}$, and $\rho \mathrm{d}t = \mathrm{d}W_x(t)\mathrm{d}W_v(t)$.
- $X_{\mathcal{P}}(t)$ is assumed to be independent of the Brownian motions and the jump sizes.
- Under this model, the variance process follows the non-central chi-square distribution, $\chi^2(\delta, \bar{\kappa}(\cdot,\cdot))$, with δ degrees of freedom and non-centrality parameter $\bar{\kappa}(t_0, t)$,

$$v(t)|v(t_0) \sim \bar{c}(t_0, t)\chi^2(\delta, \bar{\kappa}(t_0, t)),$$

  where

$$\bar{c}(t_0, t) = \frac{\gamma^2}{4\kappa}(1 - \mathrm{e}^{-\kappa(t-t_0)}), \quad \delta = \frac{4\kappa\bar{v}}{\gamma^2}, \quad \bar{\kappa}(t_0, t) = \frac{4\kappa \mathrm{e}^{-\kappa(t-t_0)}v(t_0)}{\gamma^2(1 - \mathrm{e}^{-\kappa(t-t_0)})}.$$

Implied Volatility Surface for r-Bates Model

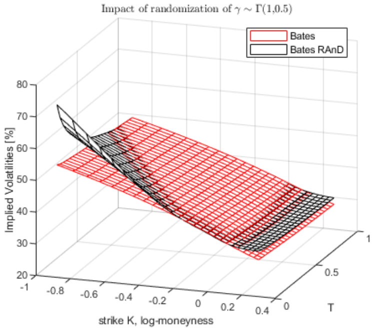

Figure: Implied volatility surface for RAnD Bates model. Left panel: randomized *vol-vol*, $\gamma \sim \Gamma(1, 0.5)$. Right panel: randomized jump's mean, $\mu_J \sim \mathcal{N}(-0.1, 0.2)$. Other model parameters are $r = 0$, $\mu_J = -0.1$, $\sigma_J = 0.06$, $\lambda = 0.08$, $\kappa = 0.5$, $\gamma = 0.5$, $\bar{v} = 0.13$, $\rho = -0.7$, $T = 1/12$, and $v_0 = 0.13$.

Random Parameters and Impact on IVs

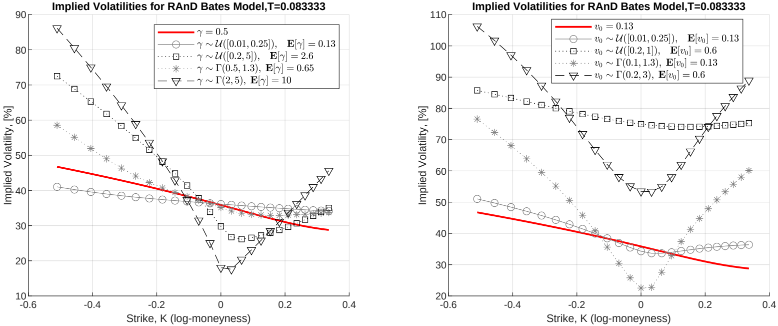

Figure: Impact of randomized parameters on implied volatilities. Left: randomized *vol-vol*, γ. Right: randomized *initial vol*, v_0.

Random Parameters and Impact on IVs

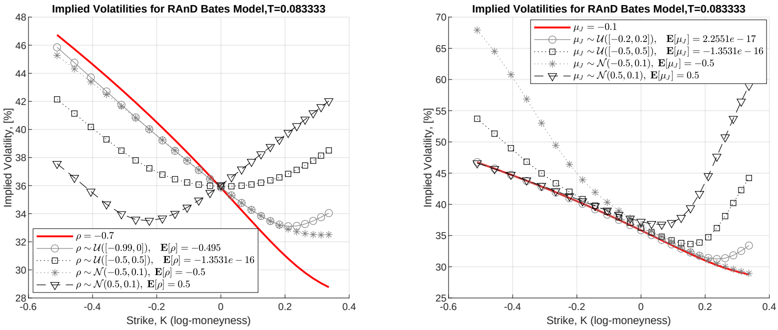

Figure: Impact of randomized parameters on implied volatilities. Left: randomized *correlation*, ρ. Right: randomized *jump's mean*, μ_J.

Pricing of Options on VIX

- For a given fixed time-horizon $[t, T]$, the volatility index of an asset $S(t)$, denoted as $\overline{\mathrm{vix}}(t, T)$, is defined as:

$$\overline{\mathrm{vix}}^2(t, T) = 100^2 \times \frac{-2}{T-t}\mathbb{E}_t\left[\log \frac{S(T)}{S(t)}\right],$$

  where $\mathbb{E}_t[\cdot]$ indicates the expectation taken under under the risk-neutral measure $\mathbb{Q}$ and the natural filtration $\mathcal{F}(t)$.

- Under the Bates model, the VIX is expressed by:

$$\begin{aligned} \overline{\mathrm{vix}}^2(t, T) &= 100^2 \times \mathrm{vix}^2(t, T), \\ \mathrm{vix}^2(t, T) &= a(t, T)v(t) + b(t, T) + c, \end{aligned}$$

  with deterministic functions $a(t, T)$, $b(t, T)$ and c.

- A call option on VIX is then given as:

$$\begin{aligned} V_{\mathrm{vix}}(t) &= \mathrm{e}^{-r(T-t)}\mathbb{E}_t\left[\max(\overline{\mathrm{vix}}(T, T+\delta T) - \overline{K}, 0)\right] \\ &= 100 \times \mathrm{e}^{-r(T-t)} \int_{\mathbb{R}^+} \max(\sqrt{v} - K, 0) f_{\mathrm{vix}^2}(v; T, T+\delta T)\mathrm{d}v. \end{aligned}$$

- At this point we need to derive coefficients H_k for the COS method.

Pricing of Options on VIX

- Since we can utilize the analytically known distribution for $\overline{\mathrm{vix}}$, the pricing may also be performed by directly integrating the payoff function and employing the PDF:

$$V_{\mathrm{vix}}(t) = 100 \times 2\alpha_1 \mathrm{e}^{-r(T-t)} \int_{\overline{K}} x(x-\overline{K}) f_{\chi^2(\delta,\bar{\kappa}(t,T))}\left(\alpha_1(x^2-\alpha_2)\right) \mathrm{d}x.$$

- With one of the model parameters stochastic, the RAnD pricing equation reads:

$$V_{\mathrm{vix}}(t) = 100 \times 2\alpha_1 \mathrm{e}^{-r(T-t)} \sum_{n=1}^{N} \omega_n \int_{\overline{K}} x(x-\overline{K}) f_{\chi^2(\delta,\bar{\kappa}(t,T))}\left(\alpha_1(x^2-\alpha_2);\theta_n\right) \mathrm{d}x,$$

  where θ_n in $f_{\chi^2(\cdot,\cdot)}(\cdot;\theta_n)$ indicates a particular realization of the model parameter and ω_n corresponds to its weight.
- Once the pricing equations are known, we can perform the model calibration.

Calibration to Options on SPX and VIX

Table: Parameters determined in calibration of S&P and VIX

Calibrated RAnD Bates parameters

date	κ	v_0	$\bar{v}$	ρ	μ_J	σ_J	λ	γ
02/02/2022	0.5	0.170^2	0.23	-0.65	-0.25	0.05	0.25	$\gamma \sim \mathcal{U}([0.01, 2.3])$
13/05/2022	0.14	0.267^2	0.28	-0.8	-0.25	0.02	0.1	$\gamma \sim \mathcal{U}([0.002, 2.1])$
14/07/2022	0.5	0.250^2	0.10	-0.85	-0.25	0.05	0.15	$\gamma \sim \mathcal{U}([0.05, 1.4])$

Calibration to Options on SPX and VIX

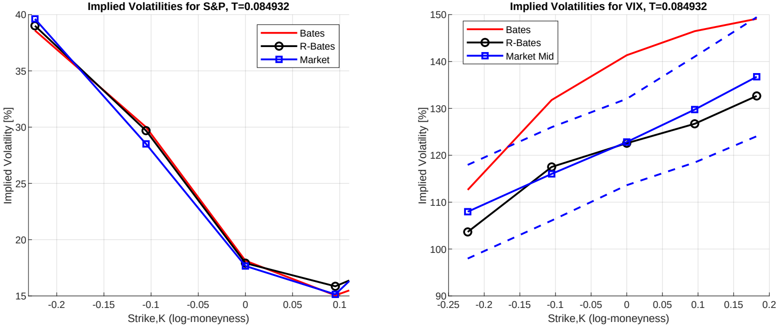

Figure: Calibration results of the RAnD Bates model. The implied volatilities for S&P and ViX were obtained on 02/02/2022. Dotted lines indicate bid-ask spreads. Left: S&P, Right: VIX.

Calibration to Options on SPX and VIX

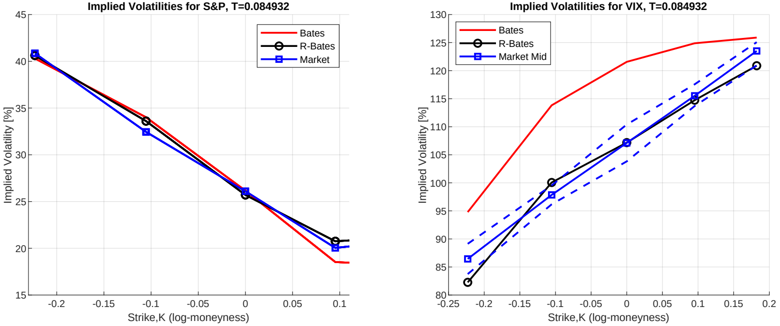

Figure: Calibration results of the RAnD Bates model. The implied volatilities for S&P and ViX were obtained on 13/05/2022. Dotted lines indicate bid-ask spreads. Left: S&P, Right: VIX.

Calibration to Options on SPX and VIX

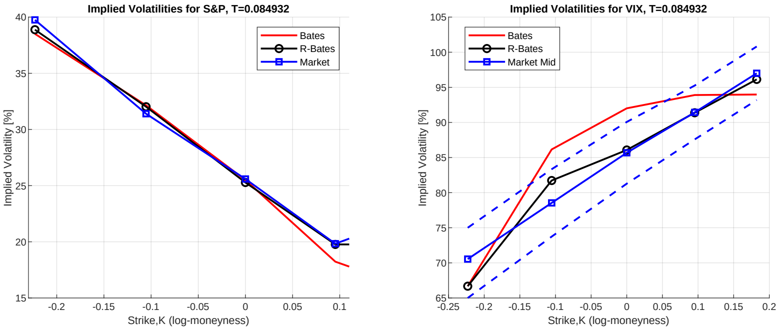

Figure: Calibration results of the RAnD Bates model. The implied volatilities for S&P and ViX were obtained on 14/07/2022. Dotted lines indicate bid-ask spreads. Left: S&P, Right: VIX.

Pricing Equations for Randomized Short-Rates

- The RAnD method can also be applied in the world of interest rates, in Grzelak (2022b), *"Randomization of Short-Rate Models, Analytic Pricing and Flexibility in Controlling Implied Volatilities"*.
- The randomization does not need to occur at the ChF level, but it can be applied to any conditional expectation.
- Consider a random variable ϑ, defined on some finite domain $D_\vartheta := [a, b]$, with its PDF, $f_\vartheta(x)$, CDF, $F_\vartheta(x)$ and a realization θ, $\vartheta(\omega) = \theta$ such that for some $N \in \mathbb{N}$ the moments are finite, $\mathbb{E}[\vartheta^{2N}] < \infty$.

$$V(t, r(t; \vartheta)) = \sum_{n=1}^{N} \omega_n V(t, r(t; \theta_n)) + \epsilon_N,$$

  where the error ϵ_N is defined as:

$$\epsilon_N = \frac{1}{(2N)!} \frac{\partial^{2N}}{\partial \xi^{2N}} V(t; r(t, \vartheta = \xi)), \quad a < \xi < b.$$

- As indicated Piterbarg (2003), averaging option prices may not be valid for non-standard options.

PDF of Randomized Models

- Under the HJM framework and the arbitrage-free condition for the drift in, the Hull-White model is specified by:

$$\gamma(t,T)=\eta\cdot\mathrm{e}^{-\lambda(T-t)},\quad t<T,$$

- We consider three different randomization cases: the randomization of the volatility parameter, η, the mean-reversion, λ, or the randomization of both parameters using bivariate distribution:

$$\eta\stackrel{\mathrm{d}}{=}\vartheta_1,\quad\text{or}\quad\lambda\stackrel{\mathrm{d}}{=}\vartheta_2,\quad\text{or}\quad\lambda|\eta\stackrel{\mathrm{d}}{=}\vartheta_2|\vartheta_1.$$

- For constant realizations of the randomized parameter, the SDE reads:

$$\mathrm{d}r(t)=\lambda(\psi(t)-r(t))\mathrm{d}t+\eta\mathrm{d}W(t),\quad r_0\equiv f(0,0),$$

  with

$$\psi(t)=f(0,t)+\frac{1}{\lambda}f(0,t)+\frac{\eta^2}{2\lambda^2}\left(1-\mathrm{e}^{-2\lambda t}\right),\quad f(0,t)=-\frac{\partial\log P(0,t)}{\partial t},$$

- As before, we can show that the PDF of the randomized HW model will be a convex sum of constituent PDFs:

$$f_{r(T)}(x)=\sum_{n=1}^{N}\omega_n f_{r(T;\theta_n)}(x)+\epsilon_N^F.$$

Dynamics of the Randomized HW model

- We consider a sequence of HW model processes, $\bar{r}_1(t), \ldots, \bar{r}_N(t)$, corresponding to parameter realizations, and the probability density relation,

$$f_{r(T)}(x) = \sum_{n=1}^{N} \omega_n f_{r(T;\theta_n)}(x).$$

- We want to determine the corresponding SDE for the rHW process, $\bar{r}(t)$. Formally, we seek an SDE, with the solution and where each of the constituent processes, $\bar{r}_n(t)$, is driven by the HW model.
- Thus, we consider the following process,

$$\mathrm{d}\bar{r}(t) = \bar{\lambda}(t, \bar{r}(t))\mathrm{d}t + \bar{\eta}(t, \bar{r}(t))\mathrm{d}W(t), \quad \bar{r}(t_0) = f(0, 0),$$

with some state-dependent drift, $\bar{\lambda}(t, \bar{r}(t))$, and volatility, $\bar{\eta}(t, \bar{r}(t))$, and where Brownian motion $W(t)$ is common for all the underlying HW processes.

Dynamics of the Randomized HW model

Proposition (Local volatility process for the HW model with randomized volatility parameter, η)

Let us assume a sequence of positive constants η_n, $n = 1, \ldots, N$. Then, the SDE

$$\mathrm{d}\bar{r}(t) = \bar{\lambda}(t, \bar{r}(t))\mathrm{d}t + \bar{\eta}(t, \bar{r}(t))\mathrm{d}W(t), \quad \bar{r}(t_0) = f(0,0),$$

with

$$\bar{\lambda}(t, y) = \sum_{n=1}^{N} \bar{\Lambda}_n(t, y)\lambda(\bar{\psi}_n(t) - y), \quad \bar{\eta}^2(t, y) = \sum_{n=1}^{N} \eta_n^2 \bar{\Lambda}_n(t, y),$$

where:

$$\bar{\Lambda}_n(t, y) = \frac{\omega_n f_{\bar{r}(t;\eta_n)}(y)}{\sum_{n=1}^{N} \omega_n f_{\bar{r}(t;\eta_n)}(y)}, \quad f_{\bar{r}(t)}(y) = \sum_{n=1}^{N} \omega_n f_{\bar{r}(t;\eta_n)}(y),$$

where $\sum_{n=1}^{N} \omega_n = 1$ for $\omega_n \geq 0$, $n = 1, \ldots, N$ with $f_{\bar{r}(t;\eta_n)}(x)$ the PDF of the HW model with dynamics, given by:

$$\mathrm{d}\bar{r}_n(t) = \lambda(\bar{\psi}_n(t) - \bar{r}_n(t))\mathrm{d}t + \eta_n \mathrm{d}W(t), \quad \bar{r}_n(t_0) = f(0,0),$$

where $\bar{r}_n(t) := \bar{r}_n(t; \eta_n)$ with $\bar{\psi}_n(t) = f(0,t) + \frac{1}{\lambda} f(0,t) + \frac{\eta_n^2}{2\lambda^2}\left(1 - \mathrm{e}^{-2\lambda t}\right)$.

Pricing of Swaptions under the rHW Model

Lemma (Pricing of Swaptions under randomized Hull-White model)

Consider the rHW model, with parameters $\{\lambda, \eta\}$ and the randomizing random variable ϑ, which randomizes either of the model parameters. For a unit notional, a constant strike, K, option expiry $T = T_{i-1}$ and a strip of swap payments $\mathcal{T} = \{T_i, \ldots, T_m\}$, with $T_i > T_{i-1}$ and accruals $\tau_i = T_i - T_{i-1}$, the prices of swaption payer and receiver, P/R := Payer/Receiver, are given by:

$$V_{P/R}^{Swpt}(t_0, T, \mathcal{T}, K; \vartheta) = \sum_{n=1}^{N} \omega_n \sum_{k=i}^{m} c_k V_{\chi}^{Z}(t_0, T, T_k, \hat{K}_k(\theta_n); \theta_n)),$$

with a swaption payer, P, for $\chi = -1$, swaption receiver, R, with $\chi = 1$, where $V_{\chi}^{Z}(\cdot)$ is the option on the ZCB and where the strike price $\hat{K}_k(\theta_n) = \exp\left(A(T, T_k; \theta_n) + B(T, T_k; \theta_n) r_n^*\right)$. Here, r_n^* is determined by solving, for each parameter realization θ_n, the following equation:

$$1 - \sum_{k=i}^{m} c_k \exp\left(A(T, T_k; \theta_n) - B(T, T_k; \theta_n) r_n^*\right) = 0, \quad n = 1, \ldots, N,$$

where $A(T, T_k; \theta_n)$ and $B(T, T_k; \theta_n)$ are known in the closed-form.

Hull-White vs. Randomized Hull-White Models

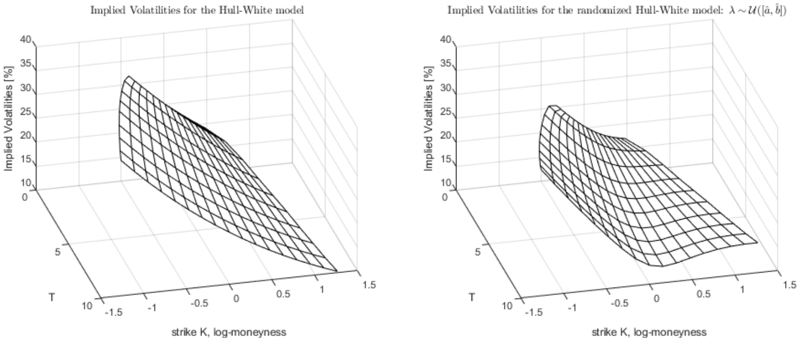

Figure: LEFT: Hull-White, RIGHT: Randomized Hull-White.
Swaption volatility evolution for the HW and rHW models implied by the shifted Black's model. The simulation was performed for varying swaption option expiry, T, and a fixed tenor of $1y$. The parameters specified in the experiment are: for the HW model: $\eta = 0.005$, $\lambda = 0.001$ and for the rHW model: $\eta = 0.005$ and $\lambda \sim \mathcal{U}([-0.15, 0.6])$. In the experiment, the implied volatilities are computed with zero shift parameter, $s = 0$.

Swaptions: Calibration Quality

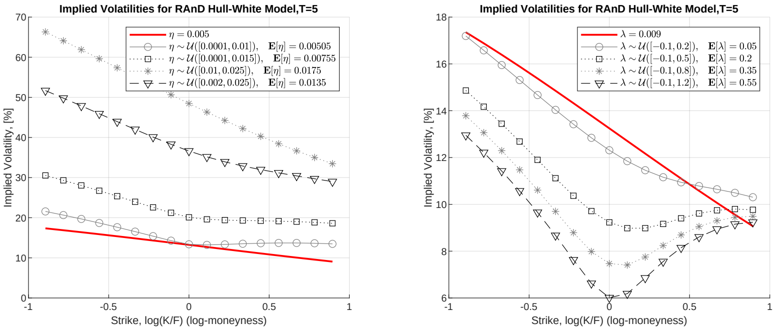

Figure: LHW: randomized volatility parameter η; RHS: randomized mean-reversion parameter λ

Swaptions: Calibration Quality

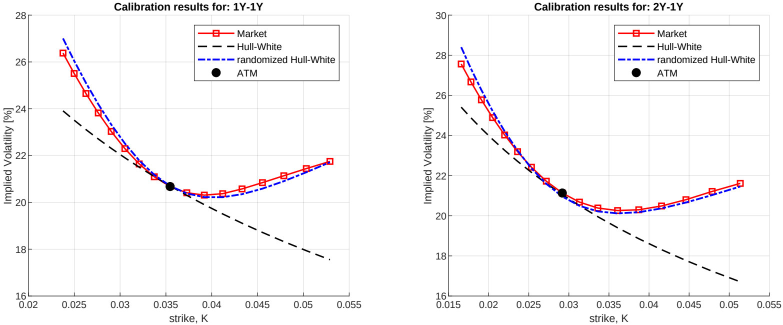

Figure: Calibration results of the HW and the rHW models. The market implied volatilities for swaptions were obtained on 18/08/2022 for the USD market. Option expiry: $T = 1y$ and $T = 2y$ and the implied volatility shift: $s = 1\%$. Calibrated parameters are presented in Table 3.

Swaptions: Calibration Quality

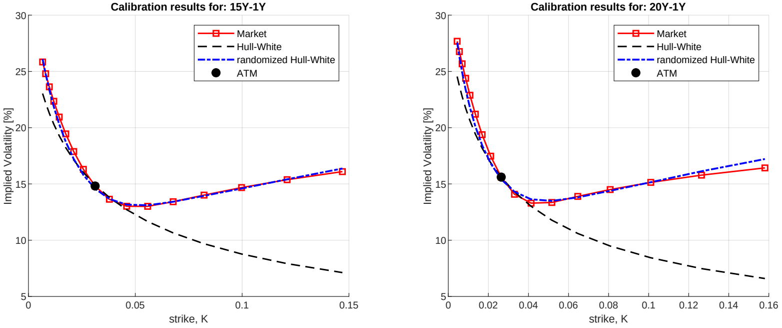

Figure: Calibration results of the HW and the rHW models. The market implied volatilities for swaptions were obtained on 18/08/2022 for the USD market. Option expiry: $T = 15y$ and $T = 20y$ and the implied volatility shift: $s = 1\%$. Calibrated parameters are presented in Table 3.

Swaptions: Calibrated Parameters

Table: Calibration of the HW and rHW model: parameters determined in swaption calibration.

	Hull-White		RAnD Hull-White	
T, expiry	η	λ	η	λ
$1y$	0.0094	0.0090	0.0091	$\lambda \sim \mathcal{N}(0.1, 0.45^2)$
$2y$	0.0082	0.0035	0.0080	$\lambda \sim \mathcal{N}(0.1, 0.33^2)$
$5y$	0.0069	0.0020	0.0079	$\lambda \sim \mathcal{N}(0.1, 0.16^2)$
$8y$	0.0067	0.0095	0.0080	$\lambda \sim \mathcal{N}(0.1, 0.12^2)$
$10y$	0.0067	0.0090	0.0082	$\lambda \sim \mathcal{N}(0.1, 0.11^2)$
$15y$	0.0064	0.0080	0.0085	$\lambda \sim \mathcal{N}(0.1, 0.09^2)$
$20y$	0.0060	0.0080	0.0086	$\lambda \sim \mathcal{N}(0.1, 0.08^2)$

- Note that the mean for λ has been fixed! Therefore the number of degrees of freedom is equal to the case for the standard Hull-White model.

PDF of Randomized Models

- We have introduced the RAnD method for efficient computation of the affine models with random parameters.
- The proposed framework is generic and can be applied to any stochastic model, even outside the class of affine diffusions.
- As long as the randomizing random variable gives rise to finite, preferably closed-form, moments, one can price European-style options efficiently.
- The heart of the method is formed by a few *critical* collocation points to recover the characteristic function.
- Fast computation of the characteristic function is possible because the method converges exponentially in the number of expansion terms.
- We have shown that the randomization of stochastic models provides a breeze of fresh air to the class of affine models.
- The application of the RAnD method to the Bates model shows that randomization allows for simultaneous calibration to S&P and VIX options-a heavily desired feature of modern models.
- Finally, we have illustrated that the model randomized Hull-White model allows for almost perfect calibration to swaption implied volatilities, while the model stays analytic and computationally efficient.

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