Principal Component Copulas for Capital Modeling

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Introduction

Popular copulas

- **Solvency II** specifies in Delegated acts that a regulatory capital model should:
  a) identify the **key variables** driving dependencies
  b) Account for **non-linear dependence** and **lack of diversification** in extreme scenarios
- **Normal mixture copulas**
  - Gaussian copula, t-copula and extensions (grouped and skewed)
  - Demarta and McNeil (2005), *The t Copula and Related Copulas*
- **Vine copulas**
  - A general copula can be decomposed in (conditional) pairwise copulae
  - Aas et al. (2009), *Pair-copula constructions of multiple dependence*
- **Factor copulas**
  - Use copula implied by factor model
  - Oh and Patton (2017), *Modeling dependence in high dimensions with factor copulas*

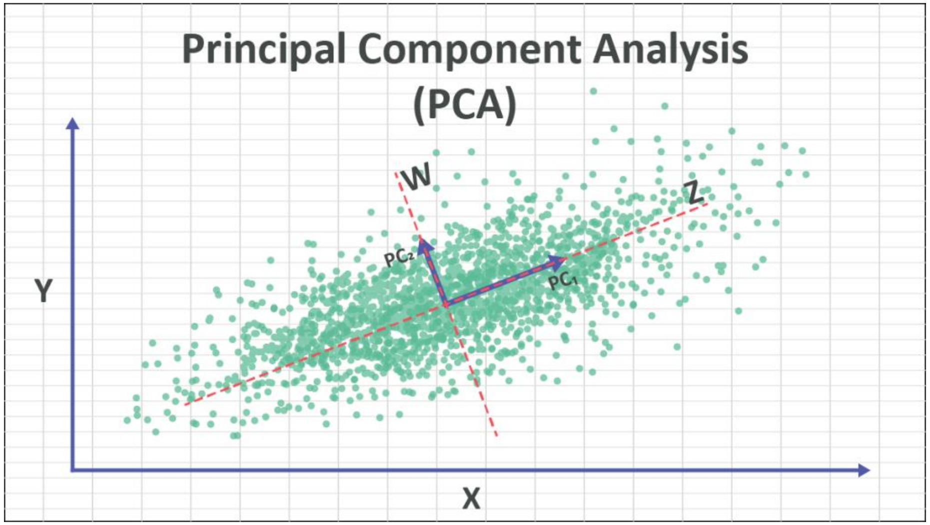
Principal Component Analysis

In market risk context

- **PCA** is a **data rotation technique**
  - Decompose correlation matrix in **eigenvalues** and **eigenvectors**
  - PCA is often used for **dimension reduction**
  - Linear relation between PCs and risk factors $\boldsymbol{Y} = \boldsymbol{P}\boldsymbol{W}^T$:

$$Y_i = \sum_{j=1}^{d} w_{j,i}^T P_j \quad \text{and} \quad P_j = \sum_{i=1}^{d} w_{i,j} Y_i$$

- PCA detects **collective market modes**
  - First PC describes the **market**
  - First eigenvalue is very large
  - First eigenvector is parallel
  - During crisis, indices move together
  - Market mode **skewed** and **fat-tailed**
  - Highest PCs describe **random noise**



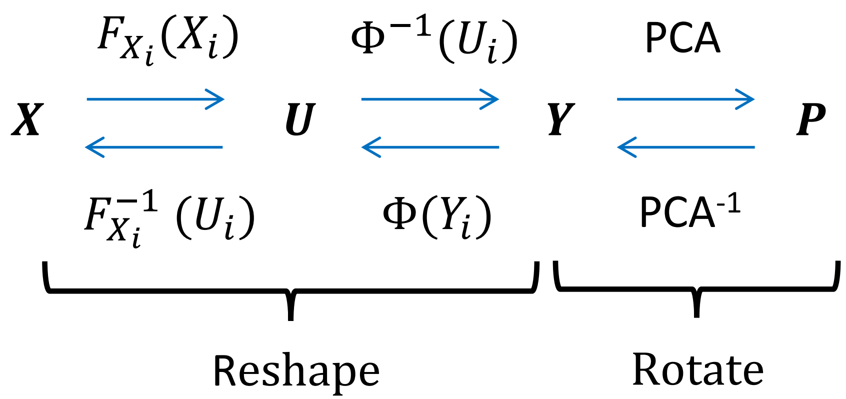
Source: https://numxl.com/

Modelling scheme

Integrating PCA into a copula framework

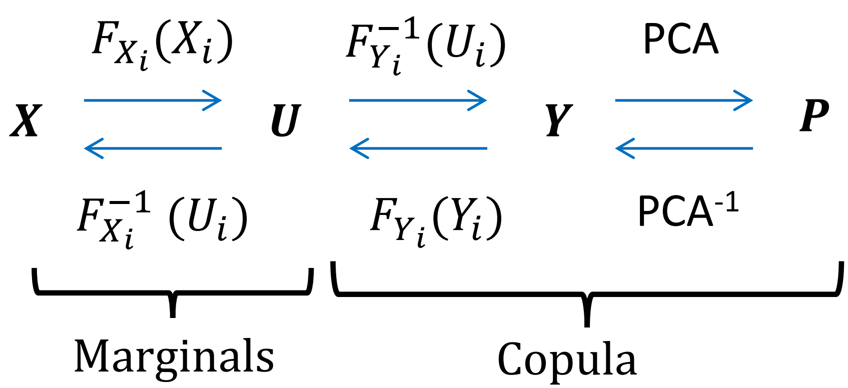
- A general integration of PCA into a copula framework is lacking

- **PCA** on **normal scores**:
  - All steps can be directly performed
  - Based on reshaping and rotating
  - Reduces skewness and outliers



- General **Principal Component Copula**
  - Defined by distribution function $F_{\boldsymbol{P}}$
  - Requires knowledge of $F_{Y_i}(Y_i)$

- An **iPCC** assumes **independent** PCs
  - Greatly simplifies the modelling
  - Normal marginals then restrictive



Propositions

Copula density and tail dependence

- **Proposition I**:
  Consider a d-dimensional iPCC with densities f_{P_j} and characteristic functions ϕ_{P_j} for PCs. Then, the copula density $c_{\boldsymbol{Y}}(u_1, \ldots, u_d)$ can be expressed in terms of f_{P_j} and ϕ_{P_j} using only **one-dimensional** integrals and inversions.

- **Proof** based on **convolution:**
  - Consider copula density: $c_{\boldsymbol{Y}}(u_1, \ldots, u_d) = \frac{f_{\boldsymbol{Y}}(y_1, \ldots, y_d)}{f_{Y_1}(y_1) \ldots f_{Y_d}(y_d)}$ with $y_i = F_{Y_i}^{-1}(u_i)$
  - Joint density $f_{\boldsymbol{Y}}$ given by: $f_{\boldsymbol{Y}}(y_1, \ldots, y_d) = f_{P_1}(\boldsymbol{y} \cdot \boldsymbol{w}_1) \ldots f_{P_d}(\boldsymbol{y} \cdot \boldsymbol{w}_d) \cdot |J|$
  - Characteristic function: $\phi_{Y_i}(t) = \prod_j \phi_{P_j}(w_{i,j} t)$
  - Marginal density: $f_{Y_i}(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{Y_i}(t) e^{-ity} dt$ (use FFT or COS)

- **Proposition II**:
  Consider two PCs with **hyperbolic** ($\propto e^{(\alpha+\beta)x}$) and **normal** distribution ($\propto e^{-x^2/(2\sigma^2)}$). Consider risk factors $Y_{\pm} = (P_1 \pm P_2)/\sqrt{2}$. Then, copula of $\boldsymbol{Y}$ has lower tail dependence parameter: $2\Phi\,(-(\alpha+\beta)\sigma)$

Estimation

Maximum likelihood and Generalized Method of Moments

- Due to analytic density **Maximum Likelihood** can be performed
- For many correlation parameters moment estimator is preferred
- Iterative algorithm for **Generalized Method of Moments**:

1. Initialize scale parameters $\hat{\rho}_{\mathbf{Y}}^{(0)}$ and shape parameters $\hat{\alpha}^{(0)}$
2. For each recursion step $k = 1, .., n_r$ update parameters as follows:
   a. Update scale parameters (correlations) given shape parameters:

   $$\hat{\rho}_{\mathrm{Y},i,j}^{(k)} = \frac{1}{N} \sum_t \hat{F}_{Y_i}^{-1} \left(u_{i,t}; \hat{\rho}_{\mathbf{Y}}^{(k-1)}, \hat{\alpha}^{(k-1)} \right) \hat{F}_{Y_j}^{-1} \left(u_{j,t}; \hat{\rho}_{\mathbf{Y}}^{(k-1)}, \hat{\alpha}^{(k-1)} \right)$$

   b. Perform PCA to update vector weights $\hat{w}_{i,j}^{(k)}$ and eigenvalues $\hat{\lambda}_j^{(k)}$
   c. Update shape parameters given scale parameters using ML:

   $$\hat{\alpha}^{(k)} = \arg\max_{\alpha} \ell(\alpha, \hat{W}^{(k)}, \hat{\Lambda}^{(k)}; u_{1,t}, .., u_{d,t}).$$

3. Return estimated GMM parameters $\{\hat{\rho}_{\mathbf{Y}}^{(n_r)}, \hat{\alpha}^{(n_r)}\}$

Simulation study

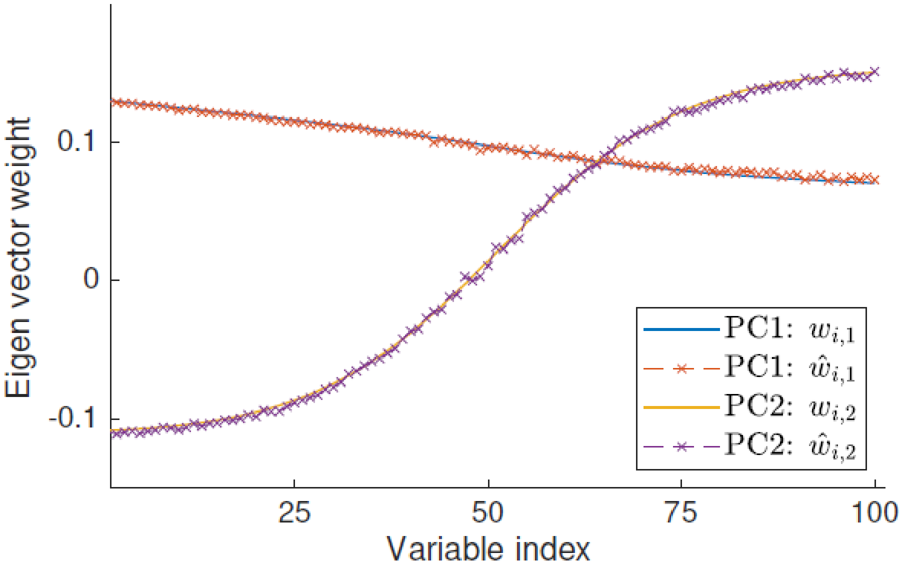
In 100 dimensions

- Data generating process (**DGP**): **Hyperbolic** for first 2 PCs, **Normal** for higher PCs
- **Correlation matrix** given by:

$$\rho_{Y_i,Y_j} = \xi_i \xi_j + \gamma_i \gamma_j \qquad \text{with} \qquad \xi_i = \frac{2}{5}\left(1 + e^{-i/d}\right), \quad \gamma_i = \frac{3}{5}\tanh(4i/d - 2).$$

- First two PCs are skewed and have kurtosis generating **asymmetric tail dependence**
- Simulation of DGP and estimation based on ML and GMM

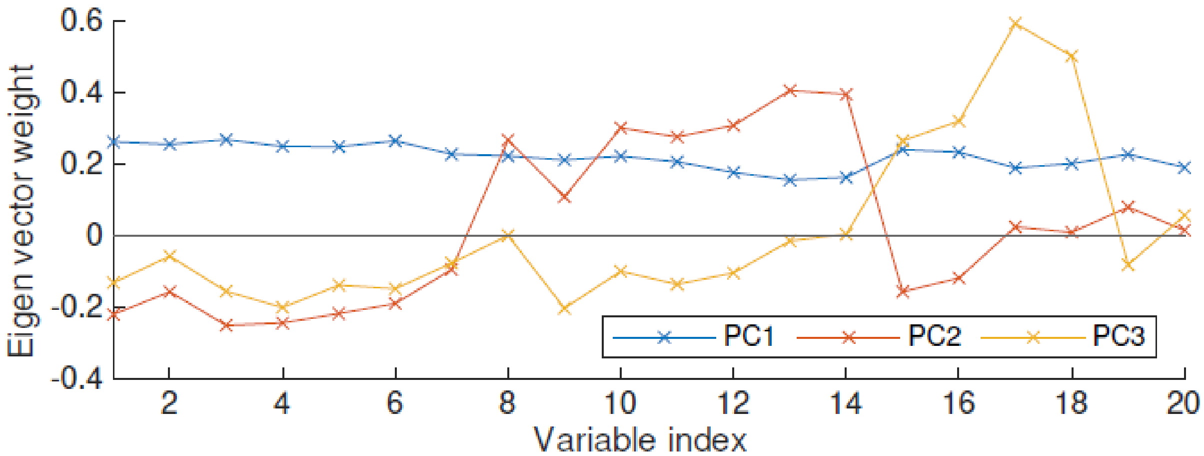
		ML		GMM	
	True	Est.	Std. Dev.	Est.	Std. Dev.
λ_1	43.78			43.67	0.70
λ_2	18.53			18.55	0.36
α_1	0.71	0.71	0.05	0.73	0.06
β_1	-0.47	-0.48	0.05	-0.49	0.05
α_2	0.94	0.95	0.12	0.94	0.13
β_2	0.31	0.32	0.08	0.31	0.09



Case study

Major stock indices

- Data used on major stock indices since 1998
- Weekly logreturn data gives 1304 data points
- GARCH (1,1)-filter for volatility clustering
- First PC describes **collective market mode** (parallel)
- Second PC describes **collective Asian mode**
- Japan (index 9) ‘most western index’



Nr.	Index	Country
1	DAX	Germany
2	FTSE 100	UK
3	CAC 40	France
4	FTSE MIB	Italy
5	IBEX 35	Spain
6	AEX	Netherlands
7	ATX	Austria
8	HSI	Hong Kong
9	Nikkei 225	Japan
10	STI	Singapore
11	Kospi	South Korea
12	Sensex	India
13	IDX Comp.	Indonesia
14	KLCI	Maleysia
15	SP 500	US
16	TSX Comp.	Canada
17	Bovespa	Brazil
18	IPC	Mexico
19	All ord.	Australia
20	TA 125	Israel

Copula performance

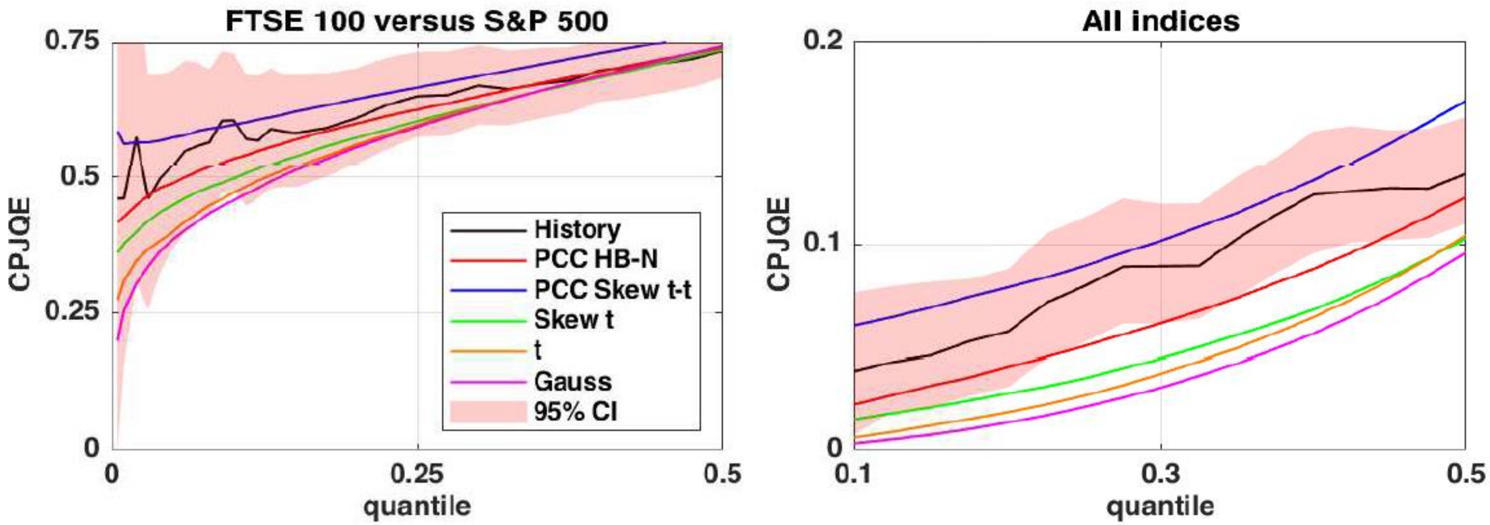
Aggregate measures of tail dependence

- We consider **Hyperbolic-Normal PCC** and **Skew *t-t* PCC**
- We study average **pairwise CPJQE** and **joint CPJQE** of equity indices:

$$\bar{\eta}_q = \frac{2}{d(d-1)} \sum_{i=1, j>i}^{d} \hat{\eta}_{q,i,j}. \qquad \text{with} \qquad \hat{\eta}_{q,i,j} = \frac{1}{nq} \sum_t I_{\{u_{i,t} \le q, u_{j,t} \le q\}}.$$

$$\hat{\eta}_q^{\text{tot}} = \frac{1}{nq} \sum_t I_{\{\hat{U}_{1,t} \le q, \hat{U}_{2,t} \le q, \ldots, \hat{U}_{d,t} \le q\}}.$$

- Conventional copulas fail **binomial tests** on **joint exceedances**, which PCC passes



Conclusions

Principal Component Copulas

- We have introduced a class of **copulas** integrating **PCA**
- With PCA we **automatically detect** most important directions in data
- The first PC describes a parallel shock, relevant for crises
- Using general distributions for first PC(s), we introduce **tail dependence**
- High-dimensional **copula density** is obtained **(semi-)analytically** using **convolution**
- The **tail dependence** properties are **analytically** obtained
- We have implemented two estimation methods, **ML and GMM**
- We have applied the model to simulations and historic data **in high dimensions**
- The model performs well on **aggregate measures of tail risk**
- We consider the copula to be particularly suited for **regulatory capital**:
  a) identify the key variables driving dependencies, and:
  b) Non-linear dependence and lack of diversification in extreme scenarios