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On the approximation of Greeks for American-style options

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Contents

2D Black–Scholes PDCP

2D Black–Scholes partial differential complementarity problem (PDCP) for two-asset American-style option value $u(s_1, s_2, t)$ at time $T - t$:

$$u \geqslant \phi, \quad \frac{\partial u}{\partial t} \geqslant \mathcal{A}u, \quad (u - \phi)\left(\frac{\partial u}{\partial t} - \mathcal{A}u\right) = 0$$

valid pointwise for (s_1, s_2, t) whenever $s_1 \geqslant 0$, $s_2 \geqslant 0$, $0 < t \leqslant T$ where

$$\mathcal{A}u = \tfrac{1}{2}\sigma_1^2 s_1^2 \frac{\partial^2 u}{\partial s_1^2} + \rho\sigma_1\sigma_2 s_1 s_2 \frac{\partial^2 u}{\partial s_1 \partial s_2} + \tfrac{1}{2}\sigma_2^2 s_2^2 \frac{\partial^2 u}{\partial s_2^2} + r s_1 \frac{\partial u}{\partial s_1} + r s_2 \frac{\partial u}{\partial s_2} - ru$$

and initial condition given by payoff

$$u(s_1, s_2, 0) = \phi(s_1, s_2)$$

for $s_1 \geqslant 0$, $s_2 \geqslant 0$.

Our main interest is in the approximation of the Greeks Delta and Gamma:

$$\Delta_1 = \frac{\partial u}{\partial s_1}\,, \quad \Delta_2 = \frac{\partial u}{\partial s_2}\,, \quad \Gamma_{11} = \frac{\partial^2 u}{\partial s_1^2}\,, \quad \Gamma_{12} = \frac{\partial^2 u}{\partial s_1 \partial s_2}\,, \quad \Gamma_{22} = \frac{\partial^2 u}{\partial s_2^2}\,.$$

They are all part of $\mathcal{A}u$.

Spatial discretization by second-order central finite differences on smooth, nonuniform, $m \times m$ grid in a truncated spatial domain $[0, S_{\max}] \times [0, S_{\max}]$ yields the semidiscrete PDCP system

$$U(t) \geqslant U_0, \quad U'(t) \geqslant AU(t), \quad (U(t) - U_0)^{\mathrm{T}}(U'(t) - AU(t)) = 0$$

for $0 < t \leqslant T$ with $U(0) = U_0$. Matrix A and vector U_0 are given.

Semidiscrete approximations to Delta and Gamma are parts of the matrix-vector product $AU(t)$.

Temporal discretization methods

Consider first the system of ODEs

$$U'(t) = AU(t) \quad (0 < t \leqslant T).$$

Parameter $\theta > 0$.

Temporal grid points $0 = t_0 < t_1 < t_2 < \ldots < t_N = T$ with step sizes $\Delta t_n = t_n - t_{n-1}$.

θ-method:

$$(I - \theta \Delta t_n A)\, U_n = U_{n-1} + (1-\theta)\Delta t_n A\, U_{n-1}.$$

For $\theta = 1$: backward Euler (BE).

For $\theta = \frac{1}{2}$: Crank–Nicolson (CN) or trapezoidal rule (TR).

Diagonally implicit Runge–Kutta (DIRK) method:

$$\begin{cases} (I - \theta \Delta t_n A)\, Y = U_{n-1} + (1-\theta)\Delta t_n A\, U_{n-1}, \\ \\ (I - \theta \Delta t_n A)\, Z = U_{n-1} + \frac{1}{2}\Delta t_n A\, U_{n-1} + \left(\frac{1}{2} - \theta\right) \Delta t_n A\, Y, \\ \\ U_n = Z. \end{cases}$$

Introduced by Cash (1984). Second-order for any θ.

A-stable whenever $\theta \geqslant \frac{1}{4}$ and L-stable if and only if $\theta = 1 \pm \frac{1}{2}\sqrt{2}$.

Independently studied for American option valuation by Khaliq, Voss & Kazmi (2006) and Ikonen & Toivanen (2007, 2009) with $\theta = 1 - \frac{1}{2}\sqrt{2}$.

Le Floc'h (2014) considered the TR-BDF2 method, which is equivalent to this.

The DIRK method is also the underlying implicit method of two well-known ADI schemes in finance: Hundsdorfer–Verwer (HV) and modified Craig–Sneyd (MCS), In 't Hout & Welfert (2007, 2009).

Adaptation of temporal discretization methods to the semidiscrete PDCP system by penalty approach, Zvan, Forsyth & Vetzal (1998, 2001), Forsyth & Vetzal (2002).

Let $Large = 10^7$ and $tol = 10^{-7}$. Set $\widehat{U}_0 = U_0$.

θ-P method:

$$\begin{cases} \left(I - \theta \Delta t_n A + P^{(k)}\right) Y^{(k+1)} = \widehat{U}_{n-1} + (1-\theta)\Delta t_n A\, \widehat{U}_{n-1} + P^{(k)} U_0 \\[2ex] \text{for } k = 0, 1, \ldots, \kappa - 1 \ \text{ and } \ \widehat{U}_n = Y^{(\kappa)} \end{cases}$$

with $Y^{(0)} = \widehat{U}_{n-1}$ and $P^{(k)}$ the diagonal matrix with i-th diagonal entry

$$P^{(k)}_{i,i} = \begin{cases} Large & \text{if } Y^{(k)}_i < U_{0,i}\,, \\ 0 & \text{otherwise.} \end{cases}$$

Convergence criterion

$$\max_i \frac{|Y_i^{(\kappa)} - Y_i^{(\kappa-1)}|}{\max\{1, |Y_i^{(\kappa)}|\}} < tol \quad \text{or} \quad P^{(\kappa)} = P^{(\kappa-1)}.$$

DIRK-P method:

$$
\begin{cases}
\left(I - \theta \Delta t_n A + P^{(k)}\right) Y^{(k+1)} = \widehat{U}_{n-1} + (1-\theta)\Delta t_n A\, \widehat{U}_{n-1} + P^{(k)} U_0 \\
\text{for } k = 0, 1, \ldots, \kappa_1 - 1 \text{ and } \widehat{Y} = Y^{(\kappa_1)}, \\
\left(I - \theta \Delta t_n A + Q^{(k)}\right) Z^{(k+1)} = \widehat{U}_{n-1} + \frac{1}{2}\Delta t_n A\, \widehat{U}_{n-1} + \left(\frac{1}{2} - \theta\right) \Delta t_n A\, \widehat{Y} + Q^{(k)} U_0 \\
\text{for } k = 0, 1, \ldots, \kappa_2 - 1 \text{ and } \widehat{U}_n = Z^{(\kappa_2)}
\end{cases}
$$

where $Z^{(0)} = \widehat{U}_{n-1}$ and $Q^{(k)}$ the diagonal matrix with i-th diagonal entry

$$
Q_{i,i}^{(k)} = \begin{cases} \textit{Large} & \text{if } Z_i^{(k)} < U_{0,i}\,, \\ 0 & \text{otherwise.} \end{cases}
$$

Iterative linear systems solver: BiCGSTAB with ILU preconditioner.

Numerical study

Four key instances:

BE-P :	θ-P method with $\theta = 1$	(first-order, L-stable)
CN-P :	θ-P method with $\theta = \frac{1}{2}$	(second-order, A-stable)
DIRKa-P :	DIRK-P method with $\theta = 1 - \frac{1}{2}\sqrt{2}$	(second-order, L-stable)
DIRKb-P :	DIRK-P method with $\theta = \frac{1}{3}$	(second-order, A-stable)

Rannacher smoothing: first two time steps of CN-P are replaced by BE-P.

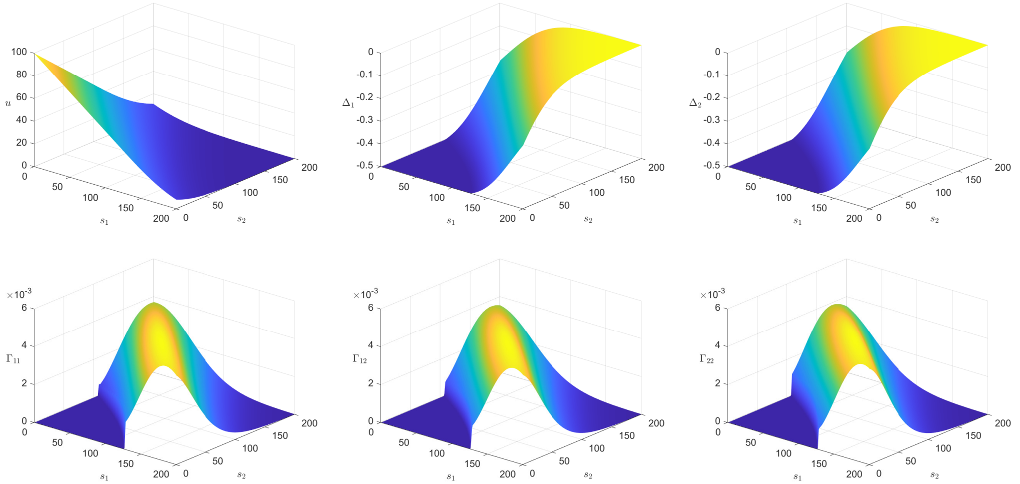
In addition, for improved convergence behaviour of second-order methods, apply nonuniform temporal grid

$$t_n = \left(\frac{n}{N}\right)^2 T \quad (n = 0, 1, 2, \dots, N),$$

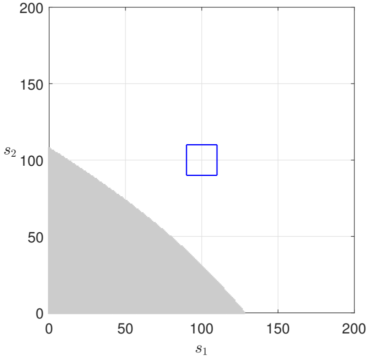
Forsyth & Vetzal (2002), Ikonen & Toivanen (2009), Reisinger & Whitley (2014).

American put-on-the-average: payoff $\phi(s_1, s_2) = \max\left(0\,,\; K - \dfrac{s_1 + s_2}{2}\right)$.

Parameter set: $\sigma_1 = 0.30,\ \sigma_2 = 0.40,\ \rho = 0.50,\ r = 0.01,\ T = 0.5,\ K = 100.$



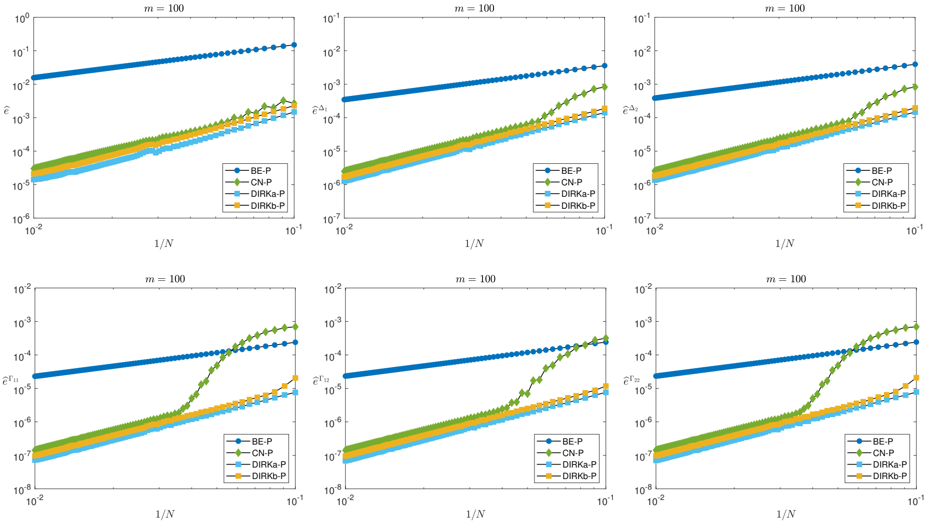
Early exercise region (grey) and region of interest (blue):



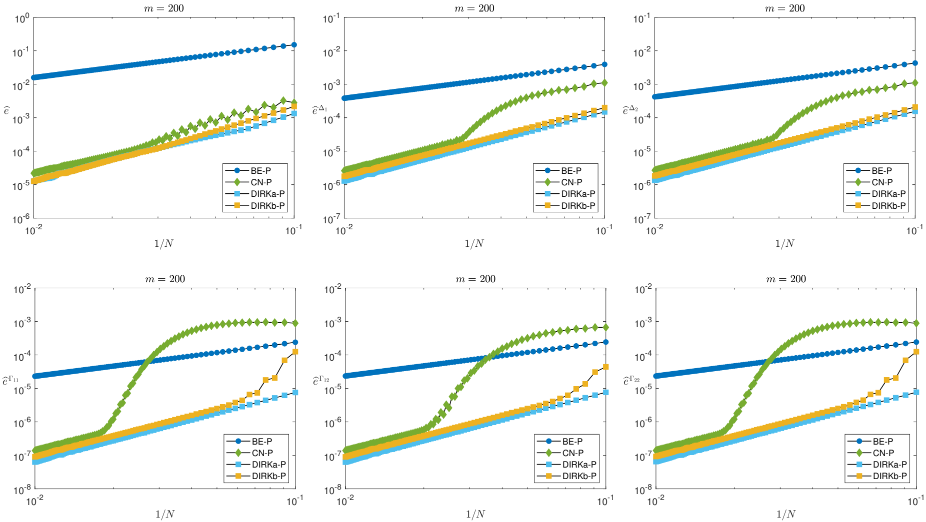
We study *temporal discretization errors* in maximum norm at $t = t_N = T$ on ROI $0.9K < s_1, s_2 < 1.1K$ for $m \times m$ spatial grids and increasing m.

Convergence in *stiff sense*: temporal error constant independent of m.

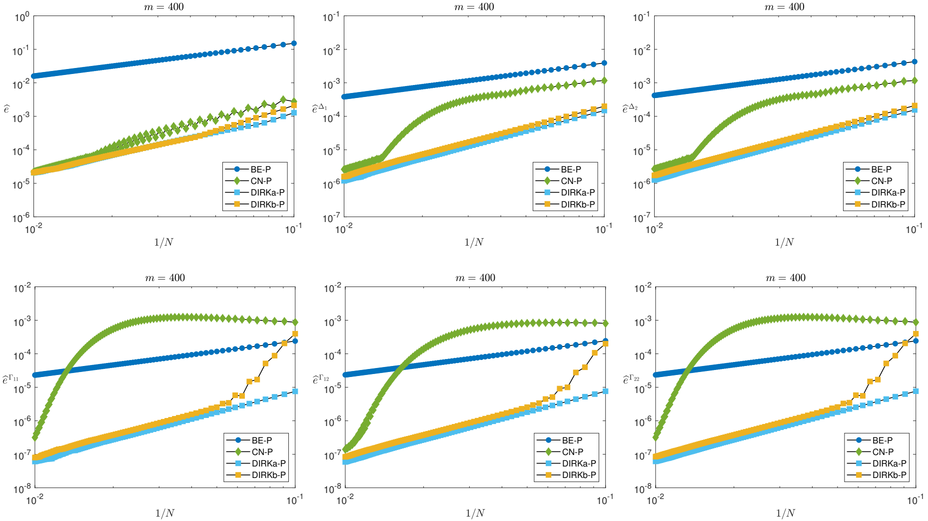
$m = 100$



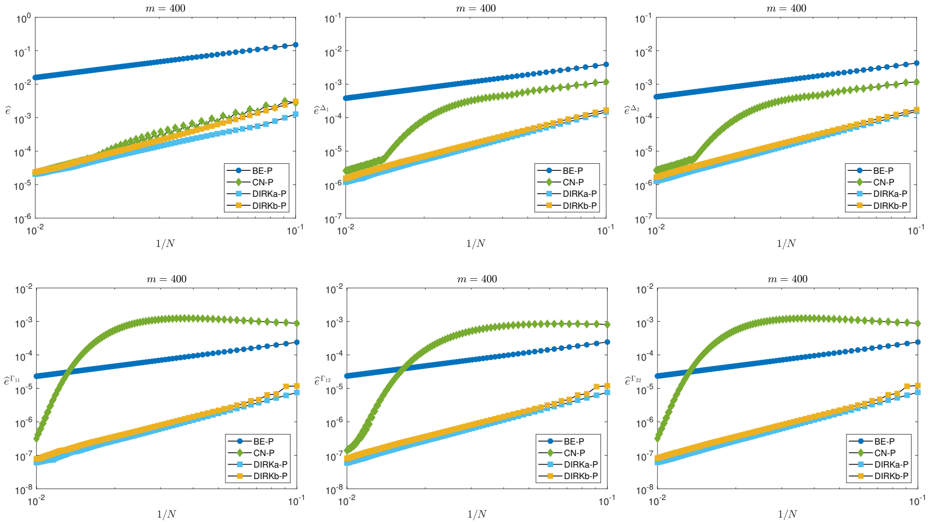
$m = 200$


$m = 200$
$\hat{e}$
$\hat{e}^{\Delta_1}$
$\hat{e}^{\Delta_2}$
$\hat{e}^{\Gamma_{11}}$
$\hat{e}^{\Gamma_{12}}$
$\hat{e}^{\Gamma_{22}}$
$1/N$
BE-P
CN-P
DIRKa-P
DIRKb-P

$m = 400$


m = 400
m = 400
m = 400
m = 400
m = 400
m = 400
1/N
1/N
1/N
1/N
1/N
1/N
BE-P
CN-P
DIRKa-P
DIRKb-P

$m = 400$ and DIRKb-P with Rannacher smoothing



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Conclusions

If ROI lies well within the continuation region, then

- BE-P method: first-order convergence in the stiff sense for the option value and all Greeks Delta and Gamma
- CN-P method: second-order convergence for Deltas and Gammas only if $N \geqslant m/\lambda$ with problem-dependent constant $\lambda > 0$ (Le Floc'h 2014, Reisinger & Whitley 2014)
- DIRKa-P method: second-order convergence in the stiff sense for the option value and all Greeks Delta and Gamma
- DIRKb-P method: second-order convergence in the stiff sense for the option value and all Greeks Delta and Gamma provided Rannacher smoothing

Reference

K.J. in 't Hout: A note on the numerical approximation of Greeks for American-style options, arXiv:2401.13361 (2024).