PDEs for pricing interest rate derivatives under the new generalized Forward Market Model (FMM)

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Outline

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IBORs scandals

- For decades, financial institutions have been using InterBank Offered Rates (IBORs) as reference rates or as underlyings of interest rate derivatives.
- At the beginning of the 21st century, several big banks manipulated the interest rate they reported that they could borrow at: IBORs scandals!
- A few years ago, financial authorities worldwide initiated the replacement of IBORs with alternative Risk Free Rates (RFRs).
- RFRs are reported to be robust because they rely on real transactions.

RFRs vs IBORs

- RFRs are overnight rates and not term rates like IBORs (i.e. one week, one month, three months, ...)
- RFRs are *backward-looking*, which means that the rate to be paid for the application period is calculated by reference to historical transaction data and set at the end of that time interval.
- IBORs are *forward-looking*, meaning that the rate to be paid for the application period is set at the beginning of that time interval.
- RFRs are risk-free since one-day credit risk can be neglected.
- RFRs not only represent the interbank market; in fact they are rates for the entire market.

LMM vs FMM

- The LIBOR Market Model (LMM) was used for the valuation of interest rate derivatives based on IBORs.
- The LMM contemplates only forward-looking rates.
- LMM it is no longer valid to price financial products based on the new RFRs, that are backward-looking.
- New mathematical models able to price the new derivatives based on RFRs:
  1. Directly simulate daily the underlying RFRs in their corresponding application periods.
  2. Models term rates based on RFRs: generalized Forward Market Model (FMM).

     Andrei Lyashenko and Fabio Mercurio, LIBOR replacement: a modelling framework for in-arrears term rates, Risk, June, 57-62, 2019.

Outline

Bank account

- A continuous-time financial market is considered.
- It has an instantaneous RFR whose value at time t is denoted by $r(t)$.
- Let $B(t)$ be the value of the bank account at time $t \geq 0$. B is the classic process that satisfies the ordinary differential equation $\mathrm{d}B(t) = r(t)B(t)\,\mathrm{d}t$ with $B(0) = 1$, so that $B(t) = e^{\int_0^t r(u)\mathrm{d}u}$.
- Risk-neutral measure $\mathcal{Q}$, whose associated numeraire is the bank account B.
- $\mathbb{E}$ will denote the expectation with respect to the risk-neutral measure.
- $\mathcal{F}_t$ will be the σ-algebra generated by risk factors up to the evaluation time.

Zero-coupon bond

- A zero-coupon bond with maturity T is a very simple contract that pays its holder one unit of currency at time T, with no intermediate payments. For $t < T$, let $P(t,T)$ be the value at time t of this product. We have the following valuation formula, which is given by risk-neutral pricing:

$$P(t,T) = \mathbb{E}\left[e^{-\int_t^T r(u)\mathrm{d}u} \middle| \mathcal{F}_t \right]. \tag{1}$$

  Note that $P(T,T) = 1$ for all T.

- Extended zero-coupon bond. For $t > T$, Equation (1) reduces to

$$P(t,T) = \mathbb{E}\left[e^{\int_T^t r(u)\mathrm{d}u} \middle| \mathcal{F}_t \right] = e^{\int_T^t r(u)\mathrm{d}u} = \frac{B(t)}{B(T)}. \tag{2}$$

  Note that $P(t,0) = B(t)$.

- The extended T-forward measure, denoted by $\mathcal{Q}^T$, is the martingale measure associated with the extended bond price $P(t,T)$. Note that the risk-neutral measure is a particular case of the extended T-forward measure where $T = 0$, i.e $\mathcal{Q} = \mathcal{Q}^0$.

The compounded setting-in-arrears term rate

- Financial derivatives written on RFRs consider as underlyings daily compounded setting-in-arrears term rates, which by definition are backward-looking in nature.

- Tenor structure $0 = T_0 < T_1 < \ldots < T_N$. Let τ_k be the year fraction of the k-th time interval $[T_{k-1}, T_k)$

- The simple backward-looking spot rate is defined as

$$R(T_{k-1}, T_k) = \frac{1}{\tau_k}\left[e^{\int_{T_{k-1}}^{T_k} r(u)\mathrm{d}u} - 1\right] = \frac{1}{\tau_k}\left[\frac{B(T_k)}{B(T_{k-1})} - 1\right] = \frac{1}{\tau_k}\left[P(T_k, T_{k-1}) - 1\right].$$

  $R(T_{k-1}, T_k)$ is the simple interest rate such that the investment of one unit of currency at time T_{k-1} yields $P(T_k, T_{k-1})$ units of currency at time T_k.

- The simple forward-looking spot rate is defined as

$$F(T_{k-1}, T_k) = \frac{1}{\tau_k}\left[\frac{1}{P(T_{k-1}, T_k)} - 1\right].$$

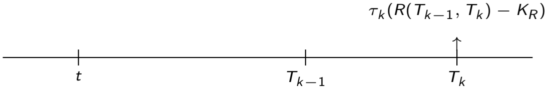
  $F(T_{k-1}, T_k)$ is the simple interest rate such that the investment of $P(T_{k-1}, T_k)$ units of currency at time T_{k-1} yields one unit of currency at time T_k.

Forward rates: Backward-looking forward rate

- The simple compounded backward-looking forward rate prevailing at time t for the time interval $[T_{k-1}, T_k)$ is denoted by $R_k(t)$ and defined by

$$R_k(t) = \frac{1}{\tau_k}\left(\frac{P(t, T_{k-1})}{P(t, T_k)} - 1\right). \qquad (3)$$

- It is the value of the fixed rate K_R in the swaplet paying $\tau_k(R(T_{k-1}, T_k) - K_R)$ at time T_k, such that this product has zero value at time t.



- Definition (3) is valid for all times t, even those times $t > T_k$.

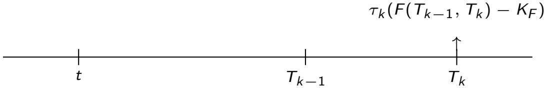
- $R_k(t)$ satisfies the following properties:
  - $R_k(T_{k-1}) = F(T_{k-1}, T_k)$, i.e., at time T_{k-1} it is equal to the forward-looking spot rate.
  - $R_k(T_k) = R(T_{k-1}, T_k)$, i.e., at time T_k it is equal to the backward-looking spot rate.
  - For $t > T_k$, $R_k(t) = R(T_{k-1}, T_k)$, i.e., after time T_k it stops evolving.

Forward rates: Forward-looking forward rate

- The simple compounded forward-looking forward rate prevailing at time t for the time interval $[T_{k-1}, T_k)$ is denoted by $F_k(t)$ and defined by

$$F_k(t) = \begin{cases} R_k(t) & \text{if} \quad t \leq T_{k-1} \\ F(T_{k-1}, T_k) & \text{if} \quad t > T_{k-1}. \end{cases} \tag{4}$$

- It is the value of the fixed rate K_F in the swaplet paying $\tau_k(R(T_{k-1}, T_k) - K_F)$ at time T_k such that this product has zero value at time t.

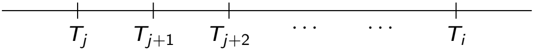


- So we have defined two types of forwards: the forward of the backward-looking rate and the forward of the forward-looking rate. Nevertheless, for each $k = 1, \ldots, N$, the backward-looking forward rate R_k and the forward-looking forward rate F_k can be modeled by a single rate, the forward of the backward-looking rate R_k.

Computation of extended discount factors from forward rates values

$P(T_i, T_j)$:

- If $T_i < T_j$, $P(T_i, T_j) = \prod_{k=i+1}^{j} \frac{1}{1 + \tau_k R_k(T_i)}$
- If $T_i = T_j$, $P(T_i, T_j) = 1$
- If $T_i > T_j$, Let us consider the scenario



From equation (3), we have

$$P(T_i, T_j) = \big(1 + \tau_{j+1} R_{j+1}(T_i)\big) P(T_i, T_{j+1}).$$

Since $T_i > T_{j+1}$, and having in mind that R_{j+1} stops evolving at time T_{j+1}, it is clear that $R_{j+1}(T_i) = R_{j+1}(T_{j+1})$. Next, by repeatedly applying (3) to the terms $P(T_i, T_{j+1})$, $P(T_i, T_{j+2})$, ... and taking into account that $R_{j+2}(T_i) = R_{j+2}(T_{j+2})$, ... and also that $P(T_j, T_j) = 1$, one readily obtains:

$$P(T_i, T_j) = \prod_{k=j+1}^{i} \big(1 + \tau_k R_k(T_k)\big). \tag{5}$$

Outline

FMM dynamics

- Model the evolution of the forward rates under a common probability measure.
- FMM dynamics under the classic spot-LIBOR measure $\mathcal{Q}^d$ and the general T_k-forward measure $\mathcal{Q}^{T_k}$ are the same as those of the corresponding LMM.
- FMM allows also for forward-rates dynamics under the risk-neutral measure $\mathcal{Q}$.
- The system of SDEs of the FMM takes the form

$$\mathrm{d}R_k(t) = \mu_k(t)\mathrm{d}t + \nu_k(t)\mathrm{d}W_k(t), \quad k = 1, \dots, N. \tag{6}$$

- The drift terms are determined by requiring lack of arbitrage.
- The diffusion terms have to capture the fact that the process $R_k(t)$ will not be killed at $t = T_{k-1}$ like it happened in the classic LMM.

FMM dynamics: diffusion terms

- Need to define dynamics of the forward rates $R_k(t)$ inside their application periods $[T_{k-1}, T_k)$.
- The volatility of $R_k(t)$ inside $[T_{k-1}, T_k)$ goes down progressively to zero: it becomes smaller and smaller until reaching the value zero at T_k.
- To model this behaviour

$$\mathrm{d}R_k(t) = \mu_k(t)\mathrm{d}t + \nu_k(t)\gamma_k(t)\mathrm{d}W_k(t), \quad k = 1, \dots, N. \tag{7}$$

- $\gamma_k(t)$ is a deterministic function to control the volatility decay.
- $$\gamma_k(t) = \begin{cases} 1 & \text{if } t \leq T_{k-1}, \\ \dfrac{T_k - t}{T_k - T_{k-1}} & \text{if } t \in (T_{k-1}, T_k), \\ 0 & \text{if } t \geq T_k. \end{cases}$$
- Classic LMM volatility

$$\nu_k(t) = \begin{cases} \sigma_k(t) & \text{normal model,} \\ \sigma_k(t)R_k(t) & \text{lognormal model,} \\ \sigma_k(t)R_k(t) + \vartheta_k & \text{shifted-lognormal model,} \\ \sigma_k(t)R_k(t)^{\beta_k} & \text{CEV model.} \end{cases}$$

FMM dynamics: drift terms

- Under the probability measure $\mathcal{Q}$ the price of the bonds $P(t, T_k)$ divided by the numeraire $B(t) = P(t, T_0)$ must be martingales. By using this condition, the drifts $\mu_k(t)$ for the forward rates can be computed starting from R_1 until R_N.
- μ_1: the process $\dfrac{P(t, T_1)}{P(t, T_0)}$ has to be martingale. By applying (3) and Ito's lemma, we get

$$\mathrm{d}\left(\frac{P(t,T_1)}{P(t,T_0)}\right) = \mathrm{d}\left(\frac{1}{1+\tau_1 R_1(t)}\right) =$$
$$\left(-\frac{\tau_1\mu_1(t)}{(1+\tau_1 R_1(t))^2} + \frac{\tau_1^2\nu_1^2(t)\gamma_1^2(t)}{(1+\tau_1 R_1(t))^3}\right)\mathrm{d}t - \frac{\tau_1\nu_1(t)\gamma_1(t)}{(1+\tau_1 R_1(t))^2}\mathrm{d}W_1(t).$$

By imposing that the drift term has to be zero to ensure the martingale property, it readily follows that

$$\mu_1(t) = \frac{\tau_1\nu_1^2(t)\gamma_1^2(t)}{1+\tau_1 R_1(t)}. \tag{8}$$

FMM dynamics: drift terms

- μ_2: the process $\dfrac{P(t, T_2)}{P(t, T_0)}$ has to be martingale. Computing

$$
\mathrm{d}\left(\frac{P(t,T_2)}{P(t,T_0)}\right) = \mathrm{d}\left(\frac{P(t,T_2)}{P(t,T_1)}\frac{P(t,T_1)}{P(t,T_0)}\right) = \mathrm{d}\left(\frac{1}{1+\tau_2 R_2(t)}\frac{1}{1+\tau_1 R_1(t)}\right) =
$$
$$
\left(-\frac{\tau_1\mu_1(t)}{(1+\tau_1R_1(t))^2(1+\tau_2R_2(t))} - \frac{\tau_2\mu_2(t)}{(1+\tau_1R_1(t))(1+\tau_2R_2(t))^2}\right.
$$
$$
+\frac{\tau_1^2\nu_1^2(t)\gamma_1^2(t)}{(1+\tau_1R_1(t))^3(1+\tau_2R_2(t))} + \frac{\tau_2^2\nu_2^2(t)\gamma_2^2(t)}{(1+\tau_1R_1(t))(1+\tau_2R_2(t))^3}
$$
$$
\left.+\frac{\tau_1\tau_2\rho_{12}\nu_1(t)\gamma_1(t)\nu_2(t)\gamma_2(t)}{(1+\tau_1R_1(t))^2(1+\tau_2R_2(t))^2}\right)\mathrm{d}t
$$
$$
-\frac{\tau_1\nu_1(t)\gamma_1(t)}{(1+\tau_1R_1(t))^2(1+\tau_2R_2(t))}\mathrm{d}W_1(t) - \frac{\tau_2\nu_2(t)\gamma_2(t)}{(1+\tau_1R_1(t))(1+\tau_2R_2(t))^2}\mathrm{d}W_1(t).
$$

Next, using (8) for μ_1 and imposing that the drift term has to be zero, we obtain

$$
\mu_2(t) = \nu_2(t)\gamma_2(t)\left(\rho_{12}\frac{\tau_1\nu_1(t)\gamma_1(t)}{1+\tau_1R_1(t)} + \frac{\tau_2\nu_2(t)\gamma_2(t)}{1+\tau_2R_2(t)}\right). \tag{9}
$$

FMM dynamics: drift terms

- μ_k: the following process has to be martingale

$$\frac{P(t,T_k)}{P(t,T_0)} = \prod_{i=1}^{k} \frac{P(t,T_i)}{P(t,T_{i-1})} = \prod_{i=1}^{k} \frac{1}{1+\tau_i R_i(t)}.$$

Using Ito's lemma, after some manipulations, one readily obtains

$$\mathrm{d}\left(\frac{P(t,T_k)}{P(t,T_0)}\right) = \prod_{j=1}^{k} \frac{1}{1+\tau_j R_j(t)} \times \Bigg[- \sum_{i=1}^{k} \nu_i(t)\gamma_i(t) \frac{\tau_i}{1+\tau_i R_i(t)} \mathrm{d}W_i(t)$$
$$\left(\sum_{i=1}^{k} \frac{\tau_i}{1+\tau_i R_i(t)} \left(-\mu_i(t) + \frac{\tau_i \nu_i^2(t)\gamma_i^2(t)}{1+\tau_i R_i(t)} \right) + \sum_{i,j=1, i<j}^{k} \rho_{ij}\nu_i(t)\gamma_i(t)\nu_j(t)\gamma_j(t) \frac{\tau_i}{1+\tau_i R_i(t)} \frac{\tau_j}{1+\tau_j R_j(t)} \right) \mathrm{d}t \Bigg].$$

Taking into account the previously computed values of $\mu_1, \ldots, \mu_{k-1}$ and imposing that the drift term has to be zero, one obtains

$$\mu_k(t) = \nu_k(t)\gamma_k(t) \sum_{i=1}^{k} \rho_{ik} \frac{\tau_i \nu_i(t)\gamma_i(t)}{1+\tau_i R_i(t)}. \qquad (10)$$

FMM dynamics: drift terms

- Since $\gamma_k(t) = 0$ for $t \geq T_k$, μ_k can be better expressed in terms of the index function
$$\eta(t) = \min\{j, 1 \leq j \leq k : T_j \geq t\}$$
which provides the index of the element in the tenor structure being not smaller than t that is the nearest to time t. Therefore, we have
$$\mu_k(t) = \nu_k(t)\gamma_k(t) \sum_{i=\eta(t)}^{k} \rho_{ik} \frac{\tau_i \nu_i(t)\gamma_i(t)}{1 + \tau_i R_i(t)}. \tag{11}$$

- All in all, the dynamics of R_k under the measure $\mathcal{Q}$ satisfy the following system of SDEs:
$$\mathrm{d}R_k(t) = \nu_k(t)\gamma_k(t) \sum_{i=\eta(t)}^{k} \rho_{ik} \frac{\tau_i \nu_i(t)\gamma_i(t)}{1 + \tau_i R_i(t)} \mathrm{d}t + \nu_k(t)\gamma_k(t)\mathrm{d}W_k(t),\ k = 1, \ldots, N. \tag{12}$$

Outline

FMM PDE

Let $\nu_k(t) = \nu_k(t, R_k(t))$ be a general instantaneous volatility for the forward rate $R_k(t)$. Under the risk-neutral measure $\mathcal{Q}$, the price of an interest rate derivative with maturity $T = T_k > T_0 = 0$ (for some $k = 1, \ldots, N$), that depends on the fixing of the rates $R_1, \ldots, R_N$, with payoff function $\varphi : [R^{min}, \infty)^N \to \mathbb{R}$, is given by

$$V(t, R_1, \ldots, R_N) = P(t, T_0)\Pi(t, R_1, \ldots, R_N), \quad t \in [T_0, T]$$

where the relative price $\Pi : [T_0, T] \times [R^{min}, \infty)^N \to \mathbb{R}$ satisfies the PDE

$$\frac{\partial \Pi}{\partial t} + \sum_{k=1}^{N} \mu_k(t) \frac{\partial \Pi}{\partial R_k} + \frac{1}{2} \sum_{k,l=\eta(t)}^{N} \rho_{kl} \nu_k(t) \gamma_k(t) \nu_l(t) \gamma_l(t) \frac{\partial^2 \Pi}{\partial R_k \partial R_l} = 0, \quad t \in [T_0, T), \tag{13}$$

along with the terminal condition

$$\Pi(T, R_1, \ldots, R_N) = \frac{\varphi(R_1, \ldots, R_N)}{P(T, T_0)}, \quad R_1, \ldots, R_N \geq R^{min}.$$

FMM PDE

PDE (13) diffuses a relative price, i.e., a price in terms of a bond. After having numerically solved the PDE and thereby having obtained the time t relative value function, the latter has to be multiplied by the time t bond price $P(t, T_0)$ to obtain the absolute value price (the price of the derivative itself). Note that if $t = T_0$, since $P(T_0, T_0) = 1$, then $V(T_0, R_1, \dots, R_N) = \Pi(T_0, R_1, \dots, R_N)$.

Outline

Numerical methods: General idea

- RFR swaptions
- Finite differences in space
- AMFR-W1 method in time: very efficient when dealing with parabolic problems involving mixed derivatives, as they avoid computing explicitly the part of the Jacobian that includes the discretization of such mixed derivatives.
- As the payoff function of the derivative that determines the dynamics of the PDE has differentiability issues near the strike values, we have explored the integration on non-uniform meshes, which contain many more points near the payoff non-differentiability area than in the rest of the domain.
- The consideration of appropriate non-uniform meshes improves the accuracy and reliability of the approximation.
- A cell averaging technique is applied to smooth the payoff at the grid points near the non-differentiability region.

Directional splitting

Initial value problem with a directional splitting

$$\begin{gathered} Y' = \mathcal{F}(t, Y) = \sum_{k=0}^{N} \mathcal{F}_k(t, Y), \quad Y(0) = Y_0, \\ \mathcal{F}_k(t, Y) = \mathcal{A}_k(t) Y, \quad k = 0, 1, \ldots, N, \\ \mathcal{A}_1(t) = \lambda_1^2(t) \tilde{\mathcal{A}}_1, \quad \mathcal{A}_k(t) = \lambda_k^2(t) \tilde{\mathcal{A}}_k^{(1)} + \lambda_k(t) \mathcal{D}_k(t) \tilde{\mathcal{A}}_k^{(2)}, \;\; k = 2, \ldots, N, \end{gathered} \tag{14}$$

where each $\mathcal{F}_k(t, Y)$ stores the components of the discretization of the advection and diffusion terms in the x_k-direction, for $k = 1, \ldots, N$, and $\mathcal{F}_0(t, Y)$ stores those of the discretization of the mixed derivatives. In this case, $\tilde{\mathcal{A}}_1$, $\{\tilde{\mathcal{A}}_k^{(1)}, \tilde{\mathcal{A}}_k^{(2)}\}_{k=2}^{N}$ are block tridiagonal constant matrices and $\mathcal{D}_k(t)$ is diagonal.
Due to the increasing stiffness of (14) as the resolution of the spatial grid increases, explicit methods are not suitable for its time integration. On the other hand, fully implicit methods requiring the computation of the exact Jacobian of the derivative function are also unsuitable because of the complicated structure of the matrix $\mathcal{A}_0(t)$.

AMFR-W1 method

For the time integration of (14) a method from the class of AMFR-W-methods is applied. In particular, we have selected the one-stage AMFR-W1 method. More precisely, given an approximation Y_n to the solution of (14) at the time $t = t_n$, this method approximates the solution at $t = t_{n+1} = t_n + \Delta t$ (with Δt being the constant step of the time discretization) by

$$\begin{aligned}
K^{(0)} &= \Delta t\, \mathcal{F}(t_n, Y_n), \\
(I - \nu \Delta t\, \mathcal{A}_k(t_n))K^{(k)} &= K^{(k-1)} + \nu(\Delta t)^2 \alpha_{k,n}, \quad k = 1, \dots, N, \\
\tilde{K}^{(0)} &= 2K^{(0)} + \theta(\Delta t)^2 G_n - (I - \theta \Delta t\, \mathcal{A}(t_n))K^{(N)}, \\
(I - \nu \Delta t\, \mathcal{A}_k(t_n))\tilde{K}^{(k)} &= \tilde{K}^{(k-1)} + \nu(\Delta t)^2 \alpha_{k,n}, \quad k = 1, \dots, N, \\
Y_{n+1} &= Y_n + \tilde{K}^{(N)},
\end{aligned} \tag{15}$$

where

$$\mathcal{A}(t_n) = \frac{\partial \mathcal{F}}{\partial Y}(t_n, Y_n) = \sum_{k=0}^{N} \mathcal{A}_k(t_n),$$

$$\alpha_{k,n} = \frac{\partial \mathcal{F}_k}{\partial t}(t_n, Y_n),\ k = 1, \dots, N, \quad G_n = \frac{\partial \mathcal{F}}{\partial t}(t_n, Y_n),$$

with parameters $\theta = 1/2$ and $\nu = \theta$ for $N = 2, 3$ and $\nu = \kappa_N N \theta$ for $N \geq 4$, where the values of κ_N are given in GlezHairerHdezPerez18 and guarantee that the AMFR-W1 method is unconditionally stable on multi-dimensional linear constant coefficient PDEs with mixed derivatives.

Numerical results: lognormal model

Swaption $T_1 \times (T_2 - T_1)$			
K	Monte Carlo Confidence Interval	PDE	Impl vol
$1.2\,K_{ATM}$	$[6.569174 \times 10^{-7}, 6.705475 \times 10^{-7}]$	6.610817×10^{-7}	0.150103
$1.1\,K_{ATM}$	$[1.229203 \times 10^{-5}, 1.235655 \times 10^{-5}]$	1.230812×10^{-5}	0.150014
K_{ATM}	$[9.663654 \times 10^{-5}, 9.681989 \times 10^{-5}]$	9.666517×10^{-5}	0.150003
$0.9\,K_{ATM}$	$[3.313149 \times 10^{-4}, 3.315975 \times 10^{-4}]$	3.314849×10^{-4}	0.150035
$0.8\,K_{ATM}$	$[6.460959 \times 10^{-4}, 6.463961 \times 10^{-4}]$	6.463699×10^{-4}	0.150143
Time	$73.32\ s$	$603.82\ s,\ M_1 = M_2 = 1024$	

Swaption $T_1 \times (T_3 - T_1)$			
K	Monte Carlo Confidence Interval	PDE	Impl vol
$1.2\,K_{ATM}$	$[5.007571 \times 10^{-6}, 5.070211 \times 10^{-6}]$	5.020028×10^{-6}	0.178879
$1.1\,K_{ATM}$	$[4.532638 \times 10^{-5}, 4.552660 \times 10^{-5}]$	4.538339×10^{-5}	0.177969
K_{ATM}	$[2.361209 \times 10^{-4}, 2.365753 \times 10^{-4}]$	2.364758×10^{-4}	0.177020
$0.9\,K_{ATM}$	$[7.014066 \times 10^{-4}, 7.020817 \times 10^{-4}]$	7.014788×10^{-4}	0.176040
$0.8\,K_{ATM}$	$[1.340121 \times 10^{-3}, 1.340854 \times 10^{-3}]$	1.340742×10^{-3}	0.175032
Time	$112.94\ s$	$4316.30\ s,\ L = 256$	

Numerical results: lognormal model

Swaption $T_1 \times (T_4 - T_1)$			
K	Monte Carlo Confidence Interval	PDE	Impl vol
$1.2\,K_{ATM}$	$[9.480228 \times 10^{-6}, 9.589930 \times 10^{-6}]$	9.523646×10^{-6}	0.184582
$1.1\,K_{ATM}$	$[7.775208 \times 10^{-5}, 7.808471 \times 10^{-5}]$	7.788910×10^{-5}	0.183922
K_{ATM}	$[3.794420 \times 10^{-4}, 3.801720 \times 10^{-4}]$	3.800981×10^{-4}	0.183272
$0.9\,K_{ATM}$	$[1.094727 \times 10^{-3}, 1.095804 \times 10^{-3}]$	1.095566×10^{-3}	0.182621
$0.8\,K_{ATM}$	$[2.081112 \times 10^{-3}, 2.082289 \times 10^{-3}]$	2.082134×10^{-3}	0.181977
Time	$150.69\ s$	$23410.36\ s,\ L = 128$	

Swaption $T_1 \times (T_5 - T_1)$			
K	Monte Carlo Confidence Interval	PDE	Impl vol
$1.2\,K_{ATM}$	$[1.485427 \times 10^{-5}, 1.501782 \times 10^{-5}]$	1500055×10^{-5}	0.188628
$1.1\,K_{ATM}$	$[1.139641 \times 10^{-4}, 1.144421 \times 10^{-4}]$	1.143997×10^{-4}	0.187909
K_{ATM}	$[5.350862 \times 10^{-4}, 5.361152 \times 10^{-4}]$	5.357548×10^{-4}	0.187452
$0.9\,K_{ATM}$	$[1.515406 \times 10^{-3}, 1.516917 \times 10^{-3}]$	1.516010×10^{-3}	0.187002
$0.8\,K_{ATM}$	$[2.869551 \times 10^{-3}, 2.871208 \times 10^{-3}]$	2.870076×10^{-3}	0.186816
Time	$196.45\ s$	$77738.56\ s,\ L = 64$	

Thank you!

Thank you for your attention!