Hierarchical and Adaptive Methods for Accurate and Efficient Risk Estimation

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April 5, 2024
ICCF24

Motivation: CVA Capital Charge

- Valuation adjustments can be expressed through a series of non-linear functions of nested conditional expectations.
- Approximation by nested Monte Carlo is expensive, especially when the underlying market variables can only be approximately sampled.
- In such settings, accurate computation of associated risk measures using nested Monte Carlo simulation is computationally infeasible.

Motivation: CVA Capital Charge (Nested Simulation)

- Given market at time 0 - simulate $\mathcal{G}_H$-measurable market and credit risk factors under the physical measure P at short risk horizon $0 < H \ll 1$.
  - Given risk factors at time H, simulate instances of default τ under the risk-neutral measure Q occurring before contract maturity $T > 0$.
    - Given the (risk-neutral) market state at default time $\tau < T$, simulate random future payoffs $\pi(S_T)$ based on the asset values S_T under the risk-neutral measure.

Value-at-Risk formula:

$$\varphi = \mathsf{P}\left[\frac{\mathrm{CVA}_H}{B_H} - \mathrm{CVA}_0 > \lambda_\varphi\right],$$

$$\mathrm{CVA}_t = B_t\,\mathsf{E}^{\mathsf{Q}}\left[\chi_{t\le\tau\le T}\,\mathrm{LGD}\max\left\{\mathsf{E}^{\mathsf{Q}}\left[\,B_T^{-1}\pi(S_T)\,\middle|\,\mathcal{G}_\tau\,\right],0\right\}\middle|\,\mathcal{G}_t\right].$$

Overview

Let X, Y, Z be random variables and $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ sending $(u, y) \mapsto f(u, y)$ be Lipschitz in u and y. Consider the system

$$\begin{aligned} \varphi &= \mathrm{P}[\, U_0(Z) > \lambda_\varphi \,] \\ U_0(Z) &= \mathrm{E}[\, f(U_1(Y), Y) \,|\, Z \,] \\ U_1(Y) &= \mathrm{E}[\, X \,|\, Y \,]. \end{aligned}$$

Key features:

- Recursive approximation of nested expectations $U_0(Z)$ and $U_1(Y)$, paired with approximation of the variables X, Y and Z.
- Approximation of discontinuous observables:

$$\varphi = \mathrm{P}[\, U_0(Z) > \lambda_\varphi \,] = \mathrm{E}\left[\, \chi_{U_0(Z) > \lambda_\varphi} \,\right].$$

Nested Monte Carlo simulation has $\mathcal{O}(\varepsilon^{-5})$ cost for a root mean square error accuracy ε.

Multilevel Monte Carlo

Want to approximate

$$\mathrm{E}[\,Q\,]$$

given approximate samples $Q \approx Q_\ell$ for $\ell \in \mathbb{N}$, with

$$\begin{aligned}
\mathrm{Cost}(Q_\ell) &\propto 2^{\gamma\ell} \\
|\mathrm{E}[\,Q - Q_\ell\,]| &\propto 2^{-\alpha\ell} \\
\mathrm{Var}[\,Q - Q_\ell\,] &\propto 2^{-\beta\ell}.
\end{aligned}$$

Then, let

$$\mathrm{E}[\,\Delta_\ell Q\,] = \begin{cases} \mathrm{E}[\,Q_\ell - Q_{\ell-1}\,] & \ell > 0 \\ \mathrm{E}[\,Q_0\,] & \ell = 0, \end{cases}$$

$$\mathrm{E}[\,Q\,] \approx \mathrm{E}[\,Q_L\,] = \sum_{\ell=0}^{L} \mathrm{E}[\,\Delta_\ell Q\,].$$

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The cost of attaining root mean square error ε using multilevel Monte Carlo is of order

$$\varepsilon^{-2} \begin{cases} 1 & \beta > \gamma \\ |\log \varepsilon|^2 & \beta = \gamma \\ \varepsilon^{-(\gamma-\beta)/\alpha} & \beta < \gamma. \end{cases}$$

Unbiased Multilevel Monte Carlo [Rhee, Glynn, 2015]

$$\mathrm{E}[\,Q\,] = \sum_{\ell=0}^{\infty} \mathrm{E}[\,\Delta_\ell Q\,] = \mathrm{E}\Big[\,(\Delta_\kappa Q)\, 2^{\zeta\kappa}/C_\zeta\,\Big]$$

where κ is a random, non-negative integer with probability mass

$$\mathrm{P}[\,\kappa = \ell\,] = C_\zeta 2^{-\zeta\ell}.$$

Provided,

$$\mathrm{Cost}(Q_\ell) \propto 2^{\gamma\ell}$$

$$\mathrm{E}[\,|Q - Q_\ell|^q\,] \propto 2^{-q\beta\ell/2},$$

$(\Delta_\kappa Q)\, 2^{\zeta\kappa}/C_\zeta$ has bounded expected sampling cost and q^{th} moment when

$$\gamma < \zeta < \frac{q}{q-1}\frac{\beta}{2} \implies q < \frac{1}{1-\beta/2\zeta}.$$

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Nested Simulation

Consider now the nested pair of expectations

$$U_0 := \mathsf{E}[\, f(U_1(Y),\ Y)\,]$$
$$U_1(Y) := \mathsf{E}[\, X \,|\, Y\,]$$

Problem: Exact samples of X and Y are not available. Instead, given $Y = y$ we approximate $X \approx X_k(y)$. Similarly, $Y \approx Y_\ell$.

Solution: Combine nested 'inner' unbiased multilevel Monte Carlo estimate of $U_1(y)$, given $Y = y$, within an 'outer' unbiased multilevel Monte Carlo estimate of U_0.

Antithetic Multilevel Difference

Consider the nested pair of expectations

$$U_0 := \mathsf{E}[\, f(U_1(Y),\ Y)\,]$$

$$U_1(Y) := \mathsf{E}[\, X \mid Y\,] = \mathsf{E}\Big[\, \Delta_{\kappa_1} X(Y) 2^{\zeta_1 \kappa_1} / C_{\zeta_1} \,\Big|\, Y\,\Big]$$

$$\mathsf{P}[\, \kappa_1 = k\,] = C_{\zeta_1} 2^{-\zeta_1 k}$$

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$$\widehat{U}_{1,\ell}(Y) := \frac{1}{N_\ell} \sum_{n=1}^{N_\ell} (\Delta_{\kappa_1} X(Y) 2^{\zeta_1 \kappa_1} / C_{\zeta_1})^{(n)}, \quad N_\ell \propto 2^\ell$$

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Antithetic multilevel difference [Bourgey, De Marco, Gobet, 2020], [Bujok, Hambly, Reisinger, 2015], [Giles, Haji-Ali, 2019]:

$$\Delta_\ell f := f\Big(\widehat{U}_{1,\ell}(Y_\ell), Y_\ell\Big) - \frac{1}{2} \sum_{i=0}^{1} f\Big(\widehat{U}^{(i)}_{1,\ell-1}(Y_{\ell-1}), Y_{\ell-1}\Big),$$

where

$$\widehat{U}_{1,\ell}(y) - \frac{1}{2} \sum_{i=0}^{1} \widehat{U}^{(i)}_{1,\ell-1}(y) = 0.$$

Convergence

Theorem ([Giles, Haji-Ali, S., 2024])

Assume f is continuous and piecewise-twice differentiable, with bounded first and second derivative, and that there is $\beta > 1$ and $q > 2$ such that

$$Cost(X_k(y)) + Cost(Y_\ell) \propto 2^k + 2^\ell$$

$$\mathrm{E}[\,|X_k(Y_\ell) - X_{k-1}(Y_\ell)|^q\,] + \mathrm{E}[\,\|Y_\ell - Y_{\ell-1}\|^q\,] \propto 2^{-q\beta k/2} + 2^{-q\beta \ell/2}$$

Then, there exists $\zeta_0, \zeta_1 > 1$ such that for $\mathrm{P}[\,\kappa_0 = \ell\,] = C_{\zeta_0} 2^{-\zeta_0 \ell}$, the random variable

$$(\Delta_{\kappa_0} f)\, 2^{\zeta_0 \kappa_0} / C_{\zeta_0},$$

has bounded expected sampling cost and variance and

$$U_0 = \mathrm{E}\Big[\,(\Delta_{\kappa_0} f)\, 2^{\zeta_0 \kappa_0} / C_{\zeta_0}\,\Big].$$

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$$\varphi = \mathrm{P}[\, U_0(Z) > \lambda_\varphi \,]$$
$$U_0(z) = \mathrm{E}[\, \Delta(z) \,|\, Z = z \,]$$
$$\widehat{U}_{0,\ell}(z) = \frac{1}{N_\ell} \sum_{n=1}^{N_\ell} \Delta^{(n)}(z)$$

Key features:

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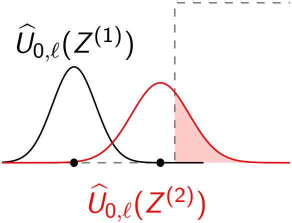
Multilevel Monte Carlo for Probabilities

$$\mathsf{P}[\,U_0(Z) > \lambda_\varphi\,] \approx \mathsf{E}[\,\chi_{\widehat{U}_{0,0}(Z)>\lambda_\varphi}\,] + \sum_{\ell=1}^{L} \mathsf{E}\Big[\,\chi_{\widehat{U}_{0,\ell}(Z)>\lambda_\varphi} - \chi_{\widehat{U}_{0,\ell-1}(Z)>\lambda_\varphi}\,\Big]$$

Theorem

For root mean square error ε and (random), positive-valued, normalising factor σ_ℓ

$$\left.\begin{array}{l} \mathsf{E}\Big[\,\Big|U_0(Z) - \widehat{U}_{0,\ell}(Z)\Big|^q \sigma_\ell^{-q}\,\Big] \propto N_\ell^{-q/2} \\ \mathsf{P}[\,|\widehat{U}_{0,\ell}(Z) - \lambda_\varphi|/\sigma_\ell \le x\,] \propto x \end{array}\right. \implies \underset{\text{Cost}}{\text{MLMC}} \propto \varepsilon^{-5/2-2/q}$$



Previous Research

- **Explicit smoothing $\chi_{x>0} \approx g(x)$:**

  [Giles, Nagapetyan, Ritter, 2015].

- **Numerical smoothing:**

  [Bayer, Ben Hammouda, Tempone, 2023], [Giles, Debrabant, Rößler, 2019].

- **Path branching:**

  [Giles, Haji-Ali, 2022].

- **Adaptivity:**
  - For partial differential equations with random coefficients [Elfverson, Hellman, Målqvist, 2016].
  - For nested expectations [Haji-Ali, S., Teckentrup, 2022], [Giles, Haji-Ali, 2019].

- **Quasi-Monte Carlo:**

  [Xu, He, Wang, 2020].

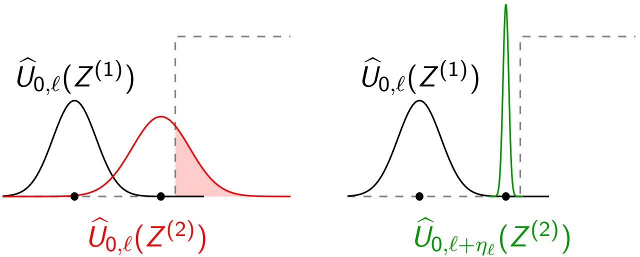
Adaptivity

$$\varphi \approx \mathrm{E}[\,\chi_{\widehat{U}_{0,0}(Z)>\lambda_\varphi}\,] + \sum_{\ell=1}^{L} \mathrm{E}\left[\chi_{\widehat{U}_{0,\ell+\eta_\ell}(Z)>\lambda_\varphi} - \chi_{\widehat{U}_{0,\ell-1+\eta_{\ell-1}}(Z)>\lambda_\varphi}\right]$$

Theorem ([Haji-Ali, S., Teckentrup, 2022])

Let η_ℓ be such that $|\widehat{U}_{0,\ell+\eta_\ell}(Z) - \lambda_\varphi| \geq \sigma_{\ell+\eta_\ell} N_\ell^{(1-r)/r} N_{\eta_\ell}^{-1/r}$ or $\eta_\ell = \ell$. Then, for a root-mean-square error $\varepsilon > 0$

$$\mathrm{E}\left[\left|U_0(Z) - \widehat{U}_{0,\ell}(Z)\right|^q \sigma_\ell^{-q}\right] \propto N_\ell^{-q/2}$$
$$\mathrm{P}[\,|\widehat{U}_{0,\ell}(Z) - \lambda_\varphi|/\sigma_\ell \leq x\,] \propto x$$
$$\implies \text{Adaptive MLMC Cost} \propto \varepsilon^{-2-2/q}$$



CVA [Giles, Haji-Ali, S., 2024]

$$\varphi = \mathsf{P}\left[\, \frac{\mathrm{CVA}_H}{B_H} - \mathrm{CVA}_0 > \lambda_\varphi \,\right],$$

$$\mathrm{CVA}_t = B_t\, \mathsf{E}^{\mathsf{Q}}\left[\, \chi_{t \le \tau \le T}\, \mathrm{LGD}\, \max\left\{ \mathsf{E}^{\mathsf{Q}}\left[\, B_T^{-1} \pi(S_T) \,\middle|\, \mathcal{G}_\tau \,\right], 0 \right\} \,\middle|\, \mathcal{G}_t \,\right].$$

Using a combination of

- Milstein simulation of the assets S_T
- Nested multilevel Monte Carlo estimation
- Unbiased multilevel Monte Carlo sampling

we can express

$$\frac{\mathrm{CVA}_H}{B_H} - \mathrm{CVA}_0 = \mathsf{E}[\, \Delta \mid Z \,],$$

where Z captures all relevant $\mathcal{G}_H$-measurable risk-factors and Δ is a random variable which can be sampled exactly.

CVA [Giles, Haji-Ali, S., 2024]

$$\varphi = \mathrm{P}[\, U_0(Z) > \lambda_\varphi \,]$$
$$U_0(Z) := \mathrm{E}[\, \Delta \mid Z \,]$$

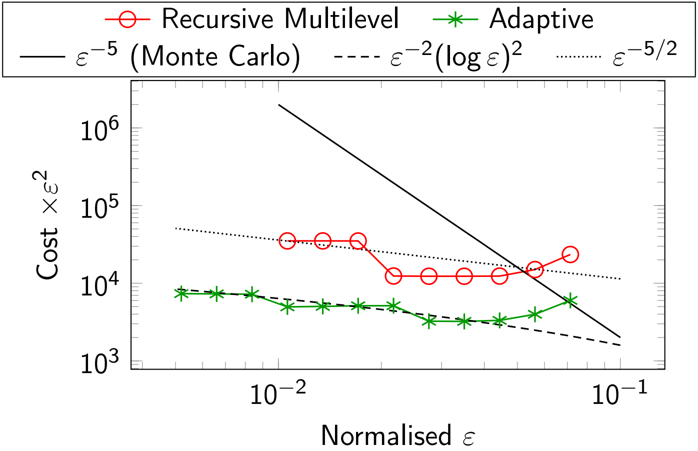
- Approximate

$$U_0(Z) \approx \widehat{U}_{0,\ell}(Z) = \frac{1}{N_\ell} \sum_{n=1}^{N_\ell} \Delta^{(n)}(Z).$$

- Adaptively add more independent samples according to the value of $|\widehat{U}_{0,\ell}(Z)|/\sigma_\ell$, where σ_ℓ^2 is the conditional sample variance.

CVA: Numerical Results

$$\varphi = \mathrm{P}[\, U_0(Z) > \lambda_\varphi \,]$$



Conclusion

- Multilevel Monte Carlo estimators can be nested to compute risk-measures involving the CVA, and are significantly more efficient than nested (single-level) Monte Carlo.
- Adaptive Sampling can be used to improve efficiency when approximating random variables about a discontinuity threshold.

Current/Future work:

- Combination of methods for solving $\varphi = \mathrm{P}[\, U_0(Z) > \lambda_\varphi \,]$ with multilevel stochastic approximation techniques to find the quantile λ_φ.
- Nested biased multilevel estimators to relax convergence assumptions.
- More complex models of CVA, including posting of collateral and more advanced models of default times.

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