Stochastic Volterra rough volatility models and Markovian approximations

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Outline

Motivation

- Empirical results of volatility time series (Gatheral *et al.* , 2018) indicate that increments of log-volatility behave like those of a fractional Brownian motion (fBm):

$$\log \sigma_{t+\Delta_t} - \log \sigma_t \propto Z^H_{t+\Delta_t} - Z^H_t. \tag{1}$$

- Observed VIX smile is upward slopping, but the rBergomi model produces flat smiles.

- The VIX index measures market expected volatility ("fear" index) and is defined by

$$VIX_T := \sqrt{\frac{1}{\Delta}\mathbb{E}\left[\int_T^{T+\Delta} v_u \, du \mid \mathcal{F}_T\right]}, \tag{2}$$

with $\Delta = 30$ days.

Motivation

- Consider Stochastic Volterra models introduced by (Horvath *et al.* , 2020), where the vol-of-vol may depend on the volatility and/or may not be Markovian.
- If the vol-of-vol is Markovian, the numerical method of least squares Monte-Carlo (LSMC) can be applied for VIX option pricing.
- Contribute to the solution of the joint calibration problem: find a stochastic volatility model with realistic dynamics that fits both the SP500 and VIX implied volatility smiles.
- In the case of non-Markovian vol-of-vol, we explore the application of a Markovian approximation of the vol-of-vol process.

The rBergomi model

- Model the variance as

$$v_u = A_0(u) \exp\left(2\sqrt{\gamma}\int_0^u K_\alpha(u-s)dW_s\right) \tag{3}$$

where A_0 is a deterministic function, γ is a positive constant and K_α is the fractional kernel

$$K_\alpha(x) = x^{\alpha-1}, \quad \alpha := H + \frac{1}{2}, \quad H \in (0, 1/2). \tag{4}$$

- The asset price follows the dynamics

$$dS_t = S_t\sqrt{v_t}d\overline{W}_t \tag{5}$$

where $\overline{W}$ and W are correlated.

Stochastic Volterra models

- A stochastic Volterra model is a natural generalization of the rBergomi model, where we replace the vol-of-vol γ by a stochastic process Γ.

- More precisely:

$$v_u = A_0(u) \exp(2X_u), \tag{6}$$

where A_0 is a deterministic function and X is a truncated Brownian semi-stationary process (TBSS) given by

$$X_u = \int_0^u \sqrt{\Gamma_s} K_\alpha(u-s)\, dW_s. \tag{7}$$

Representation of SVM

Proposition

For $u \geq t \geq 0$, let

$$A_t(u) := \frac{\xi_t(u)}{h_t(u)}, \tag{8}$$

where

$$h_t(u) = \mathbb{E}\left[E_{t,u}(u) \mid \mathcal{F}_t\right] \tag{9}$$

and

$$E_{p,q}(u) = \exp\left(2\int_p^q K_\alpha(u-s)\sqrt{\Gamma_s}\,\mathrm{d}W_s\right). \tag{10}$$

Then we have the time-invariant decomposition

$$v_u = A_0(u)E_{0,u}(u) = A_t(u)E_{t,u}(u). \tag{11}$$

VIX option pricing

- The "hard" part of VIX pricing is the curve h_T. Indeed,

$$\Delta VIX_T^2 = \int_T^{T+\Delta} \xi_T(u)\,\mathrm{d}u = \int_T^{T+\Delta} \frac{\xi_0(u)}{h_0(u)} E_{0,T}(u) h_T(u)\,\mathrm{d}u. \quad (12)$$

- $E_{0,T}(u)$ can be computed simply by using Riemann sums. ξ_0 can be observed from the market or assumed constant (one more parameter of the model). As for h_0, it can be written as follows:

$$h_0(u) = \mathbb{E}\left[E_{0,T}(u) h_T(u)\right].$$

Least Squares Monte Carlo

- Take the infinite dimensional state variable h_T. Then, fix a time grid $(u_j)_{j=1}^n$ between T and $T+\Delta$ and obtain the projection $Y = (h_T(u_j))_{j=1}^n$.
- We use the Least squares Monte Carlo (LSMC) method - (Longstaff & Schwartz, 2001) - to approximate Y. Since Γ is Markovian, we have

$$Y_j = h_T(u_j) = \mathbb{E}\left[2\int_T^{u_j} \sqrt{\Gamma_s} K_\alpha(u_j - s)\,\mathrm{d}W_s \mid \Gamma_T\right] = f_j(\Gamma_T), \quad (13)$$

  for some deterministic function f_j.
- Write this as a classical multivariate regression problem:

$$Y = f(\Gamma_T) = \tilde{f}(\Gamma_T) + \varepsilon, \quad (14)$$

  where $\tilde{f}$ is the deterministic approximation function and ε is the stochastic error term.

Least Squares Monte Carlo

- Start by producing $\Gamma_T^{(i)}$, $i = 1, ..., K$ outer simulations.

- For each simulation of a sample of N outer simulations ($N < K$), produce M inner simulations and compute an estimate y_j of $\mathbb{E}\left[L \mid \Gamma_T^{(j)}\right]$, $j = 1, ..., N$.

- Fit a regression model $\tilde{f}$ with $\Gamma_T^{(j)}$ as predictors and y_j as target. A typical regression model is a linear regression on Hermite polynomials

$$\tilde{f}(x) = \sum \alpha_k H_k(x). \tag{15}$$

- Use the regression to obtain $Y_i = \mathbb{E}\left[L \mid \mathcal{F}_T\right]$ based on Γ_T^i (using $\tilde{f}(\Gamma)$). For more details, see (Guerreiro & Guerra, 2022).

CIR stochastic Volterra model

- Let us assume that the vol-of-vol is modeled by a CIR process:

$$d\Gamma_t = \kappa(\theta - \Gamma_t)\,\mathrm{d}t + \delta\sqrt{\Gamma_t}\mathrm{d}B_t, \tag{16}$$

where $\theta, \kappa, \delta > 0$ and B is a sBm correlated with W and $\overline{W}$ with correlation matrix for $(\overline{W}, W, B)$:

$$\Sigma = \begin{bmatrix} 1 & \rho & \rho_S \\ \rho & 1 & \rho_V \\ \rho_S & \rho_V & 1 \end{bmatrix}.$$

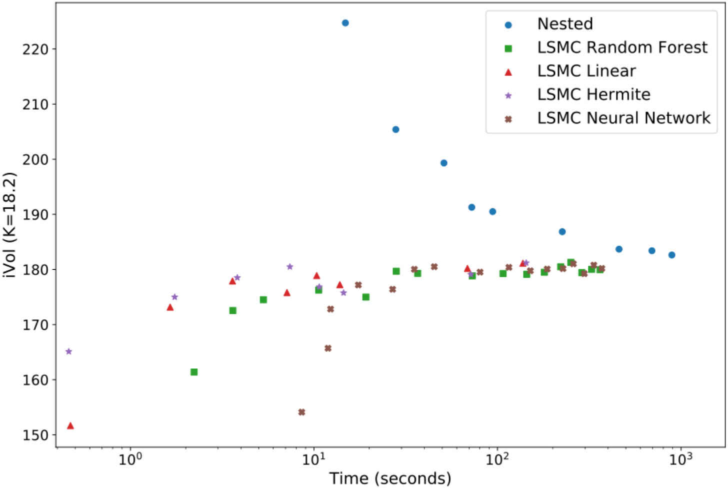

Figure: Implied Volatility Convergence - Monte-Carlo method

Calibrated S&P500 smile

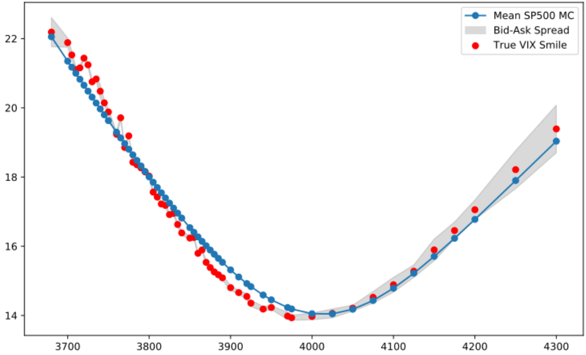

Figure: Calibrated S&P500 smile

Calibrated VIX smile

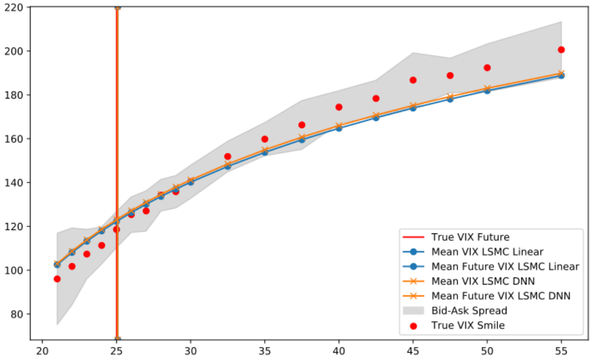

Figure: Calibrated VIX smile

Calibrated parameters

Parameter	Value
ρ_S	-0.974772
ρ_V	0.972984
δ	1.643538
γ	0.137467
H	0.063667
κ	48.868527
$\xi_0^{(1)}$	0.058979
$\xi_0^{(2)}$	0.072898
ρ	-0.90092
θ	0.037869

Table: Calibrated Values

Rough vol-of-vol

- The SVM framework is able to account for rough vol-of-vol. This has been suggested by high-frequency data analysis for major volatility indexes (Da Fonseca & Zhang, 2019): log of vol-of-vol can be modeled by a fBm with $H < 1/2$.

- The LSMC approach is not well suited for rough vol-of-vol models.

- We consider Markovian approximations for rough vol-of-vol models. These Markovian approximations are able to produce processes which are rough-like processes remaining analytically tractable.

Markovian approximations

- Stochastic rough Volterra equation:

$$X_t = x_0 + \int_0^t K(t-s)b(X_s)ds + \int_0^t K(t-s)\sigma(X_s)dW_s, \qquad (17)$$

where $x_0 \in \mathbb{R}$, b and σ are globally Lipschitz functions and the kernel $K \in L^2_{loc}(\mathbb{R}^+)$.

- Any completely monotone kernel (such as the fractional kernel) can be approximated by

$$\tilde{K}(t) = \sum_{i=1}^{n} w_i e^{-c_i t} \qquad (18)$$

for some weights $w_1, ..., w_n \in \mathbb{R}^+$ and points $c_1, ..., c_n \in \mathbb{R}_0^+$ (see (Alfonsi & Kebaier, 2021)).

Markovian approximations

Proposition

Suppose $m_1, ..., m_n \in \mathbb{R}$ are such that $\sum_i w_i m_i = x_0$. Then the solution of

$$\tilde{X}_t = x_0 + \int_0^t \tilde{K}(t-s)b(X_s)ds + \int_0^t \tilde{K}(t-s)\sigma(X_s)dW_s. \tag{19}$$

is given by

$$\tilde{X}_t = \sum_{i=1}^n w_i X_t^i, \tag{20}$$

where X^i solves the SDE

$$dX_t^i = c_i(m_i - X_t^i)dt + b\left(\sum_{j=1}^n w_j X^j\right)dt + \sigma\left(\sum_{j=1}^n w_j X^j\right)dW_t, \tag{21}$$

$$X_0^i = m_i.$$

Rough CIR vol-of-vol model

- Let us model the rough vol-of-vol as a rough CIR process:

$$X_t = \int_0^t K_\alpha(t-s)\kappa(\mu - X_s)ds + \int_0^t K_\alpha(t-s)\nu\sqrt{X_s}dZ_s, \tag{22}$$

where κ, μ, ν are positive constants.

- Markovian approximation of X:

$$\Gamma_s = \sum_{i=1}^n w_i X_s^i, \tag{23}$$

where

$$dX^i = c_i(m_i - X^i)ds + \kappa(\mu - \sum_{j=1}^n w_j X^j)ds + \nu\sqrt{\sum w_j X^j}dZ, X_0^i = m_i, \tag{24}$$

$$\sum_{i=1}^n w_i m_i = \Gamma_0 > 0. \tag{25}$$

Rough CIR vol-of-vol

Proposition

Suppose there exists j such that

$$\zeta := \sum_{i=1}^{n} w_i(c_i m_i + \kappa\mu) > \frac{1}{2}\nu^2 w_j^2. \tag{26}$$

Then the SDE (24) *has a unique continuous strong solution such that* $\Gamma_t = \sum w_i X_t^i > 0$ *for all* $t \geq 0$ *a.s.*

Rough CIR vol-of-vol

- Suppose that Z is independent of W so that Γ is independent of W. Then, we have

$$h_t(u) = \mathbb{E}\left[\exp\left(2\int_t^u \Gamma_s K^2(u-s)\,ds\right)\,\middle|\,\mathcal{F}_t\right]. \tag{27}$$

- We wish to investigate if we can obtain an exponentially affine formula for h_t. We will consider the process

$$M_s = \exp\left(2\int_t^s \Gamma_r K^2(u-r)\,dr\right)\exp\left(\varphi(u-s) + \sum_{i=1}^n w_i\psi_i(u-s)X_s^i\right) \tag{28}$$

for $t \le s \le u$. Functions $(\varphi, \psi_1, ..., \psi_n)$ will be constructed so that M is a local martingale.

Local martingale

Applying Itô formula to M, we obtain:

Proposition

Let $0 < t \leq u$ and $(X^1, ..., X^n)$ solve SDE (24). Suppose $(y_1, ..., y_n)$ solves the ODE

$$y_i' = -c_i(y_i+2G)+R((y_1+2G, ..., y_n+2G)) = -c_i\psi_i+R(\psi_1, ..., \psi_n), y_i(0) = 0 \tag{29}$$

where $R(z_1, ..., z_n) := -\kappa \sum_{i=1}^n w_i z_i + \frac{1}{2}\nu^2 \sum_{1\leq i,j\leq n} w_i w_j z_i z_j$ and $G(t) := \int_0^t K_\alpha^2(s)\, ds$. Define the functions ψ_i by $\psi_i = y_i + 2G$ and suppose φ solves the ODE

$$\varphi' = F(\psi_1, ...\psi_n), \quad \varphi(0) = 0, \tag{30}$$

where $F(z_1, ..., z_n) := \sum_{i=1}^n w_i z_i(c_i m_i + \kappa\mu)$. Define the process M by (28) using these functions $(\varphi, \psi_1, ..., \psi_n)$. Then, M is a local martingale in $[t, u]$.

Existence of solutions for the ODE's

Proposition

Define $R_i(x_1, ...x_n) := -c_i x_i + R(x_1, ...x_n)$. *Let* $\Lambda > 0$ *and assume there exists a vector* $(a_1, .., a_n)$ *such that* $2G(\Lambda) + \Lambda \max_Q R_i \leq a_i$, *where* Q *is the set* $Q = \{x \in \mathbb{R}^n \mid x_i \in \{0, a_i\}, i = 1, ..., n\}$.
Then (29) *and* (30) *have solutions on* $[0, \Lambda]$ *with bounds*

$$0 \leq \psi_i := y_i + 2G \leq a_i, \qquad 0 \leq \varphi \leq \Lambda \max_Q F. \tag{31}$$

Conclusions

- We proposed a method for VIX derivatives pricing in the generalized framework of stochastic Volterra models, by using the LSMC method
- We obtained good qualitative fits for the joint SP-VIX calibration problem.
- For the non-Markovian model of rough vol-of-vol, we considered Markovian approximations and explored the case of the rough CIR model.
- Partial results: we identified a key martingale condition to express the VIX in terms of the solution of a certain ODE and provided sufficient conditions for the existence of solutions, we verified that the process associated to an efficient VIX pricing is a local martingale.

Thank you for your attention!

Acknowledgments

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