Product of VAR time series with an application to electricity load prediction errors

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Agenda

1. Motivation – why a product random variable?
2. Considered model
   - Theoretical results
   - Illustration on simulated trajectories
3. Electricity market case study
4. Possible implications

Motivation

- Economic variables usually are modelled as separate time series
  - Prices
  - Exchange rates
  - Volumes
  - …
- In some contexts the product of variables is of the main importance, e.g.
  - Prices in local currency = prices * exchange rates
  - Transaction values = prices * volumes
  - <u>Financial cost of prediction/technical errors = price * size of the error</u>

Vector autoregression

- Vector autogresssion model (VAR(p)) is a classical approach for
  - modeling relationship between multiple quantities
  - their evolution over time

Definition 1. The bi-dimensional finite-variance VAR(1) time series $\{\mathbf{X}(t),\ t \in \mathbb{Z}\}$ satisfies the following equation

$$\mathbf{X}(t) - \Phi\mathbf{X}(t-1) = \mathbf{Z}(t), \tag{1}$$

where $\mathbf{X}(t) = (X_1(t), X_2(t))$, Φ is 2×2 matrix

$$\Phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}, \tag{2}$$

and $\{\mathbf{Z}(t),\ t \in \mathbb{Z}\}$ is the zero-mean bi-dimensional residual series, i.e. for each $t \in \mathbb{Z}$, $\mathbf{Z}(t) = (Z_1(t), Z_2(t))$.

Product of VAR components

- Introduce a product time series

$$Y(t) = X_1(t)X_2(t),$$

- If $\det(\mathbf{I} - z\Phi) \neq 0 \; \forall_{z \in Z} |z| \leq 1$, then $Y(t)$ can be written using a casual representation

$$Y(t) = \sum_{j,i=0}^{\infty} \sum_{k,l=1}^{2} \phi_{1k}^{(j)} \phi_{2l}^{(i)} Z_k(t-j) Z_l(t-i)$$

- Distribution of $Y(t)$ is related to the distribution of $Z_1 \; Z_2$

$$f_{Z_1 Z_2}(z) = \int_{-\infty}^{\infty} \frac{1}{|z_1|} f_{Z_1, Z_2}\left(z_1, \frac{z}{z_1}\right) dz_1$$

Properties of the product

$$\mathbb{E}(Y(t)) = \gamma_{X,1,2} = \sum_{j=0}^{\infty} \sum_{k,l=1}^{2} \phi_{1k}^{(j)} \phi_{2l}^{(j)} \gamma_{Z,k,l}$$

Mean is equal to the covariance of VAR(1) components

$$\mathbb{V}ar(Y(t)) = \mathbb{E}\left[\left(\sum_{j,i=0}^{\infty} \sum_{k,l=1}^{2} \phi_{1k}^{(j)} \phi_{2l}^{(i)} Z_k(t-j) Z_l(t-i)\right)^2\right] - \gamma_{X,1,2}^2$$

Distribution of $Y(t)$ is influenced by the dependence of $X_1(t)$ and $X_1(t)$

- coefficients Φ
- $\left(Z_1(t), Z_2(t)\right)$ distribution

$$ACVF_Y(t, t+h) = \sum_{j,i=0}^{\infty} \sum_{m,p=-h}^{\infty} \sum_{k,l,n,r=1}^{2} \phi_{1k}^{(j)} \phi_{2l}^{(i)} \phi_{1n}^{(m+h)} \phi_{2r}^{(p+h)} \mathbb{E}\left[Z_k(t-j) Z_l(t-i) Z_n(t-m) Z_r(t-p)\right] - \gamma_{X,1,2}^2,$$

Special cases

1. Independent components –> product of AR(1) time series

$$X_1(t) - \phi_{11} X_1(t-1) = Z_1(t)$$

$$X_2(t) - \phi_{22} X_2(t-1) = Z_2(t)$$

$$\mathbb{E}(Y(t)) = 0$$

$$\mathbb{V}\mathrm{ar}(Y(t)) = \frac{\sigma^2_{Z,1}\sigma^2_{Z,2}}{(1-\phi^2_{11})(1-\phi^2_{22})}$$

$$\mathrm{ACVF}_Y(h) = \frac{\sigma^2_{Z,1}\sigma^2_{Z,2}(\phi_{11}\phi_{22})^h}{(1-\phi^2_{11})(1-\phi^2_{22})}$$

Products of $X_1(t)$ and $X_2(t)$ moments

$\phi_{11}\phi_{22} > 0$

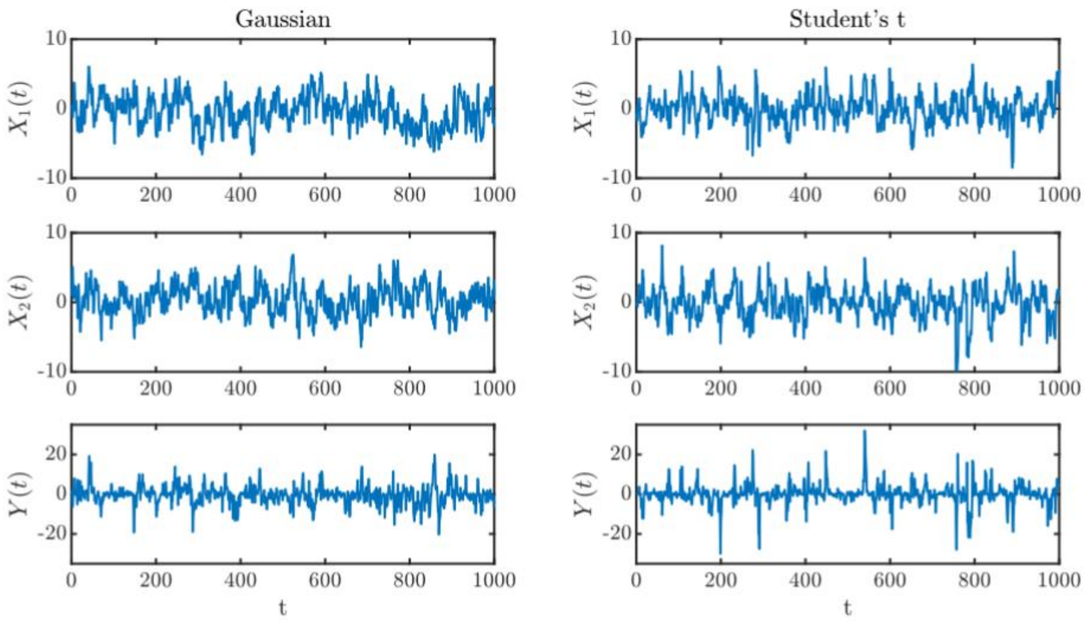


Fig. 1: Sample trajectories of the VAR(1) model components and their product for the Gaussian (left panels) and Student's t distribution (right panels). The parameters correspond to Case 1, i.e. $\phi_{11} = 0.8$, $\phi_{22} = 0.8$, $\phi_{12} = \phi_{21} = 0$ and the residual vectors $Z_i(t), i = 1, 2$ are independent with $\eta = 5$ for the Student's t distribution and $\sigma^2_{Z,1} = \sigma^2_{Z,2} = \frac{\eta}{\eta-2}$ for the Gaussian one.

$\phi_{11}\phi_{22} < 0$

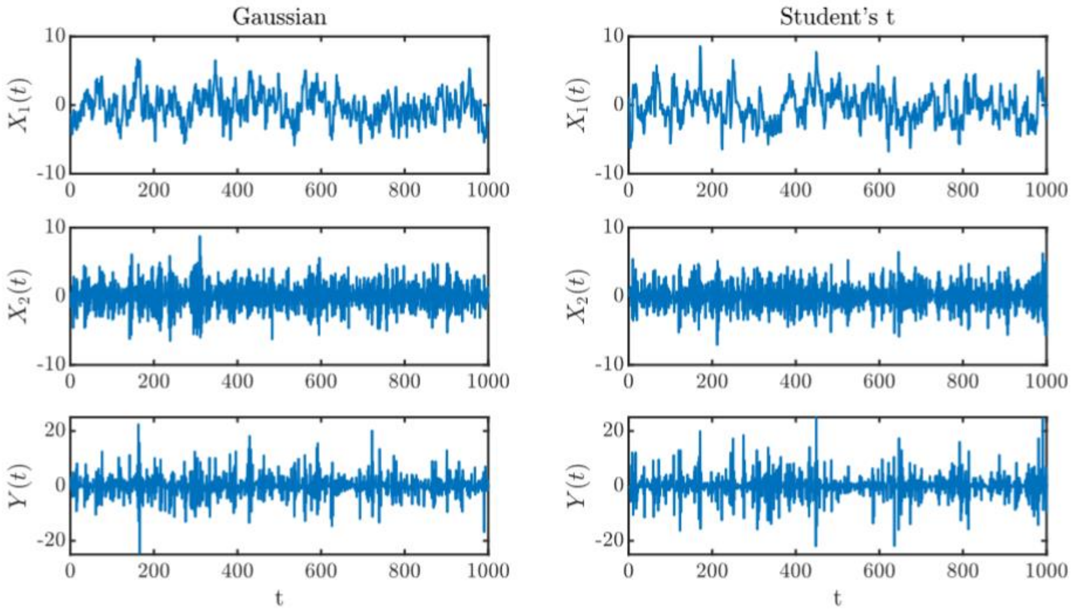
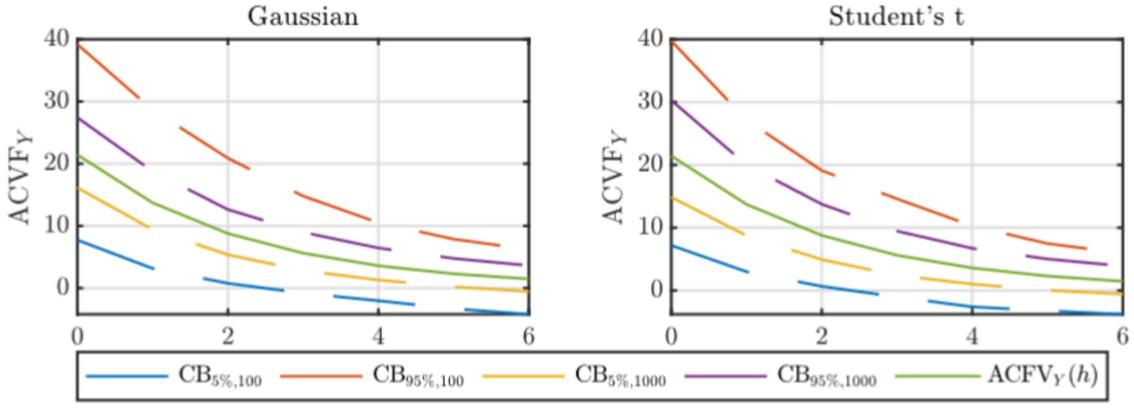
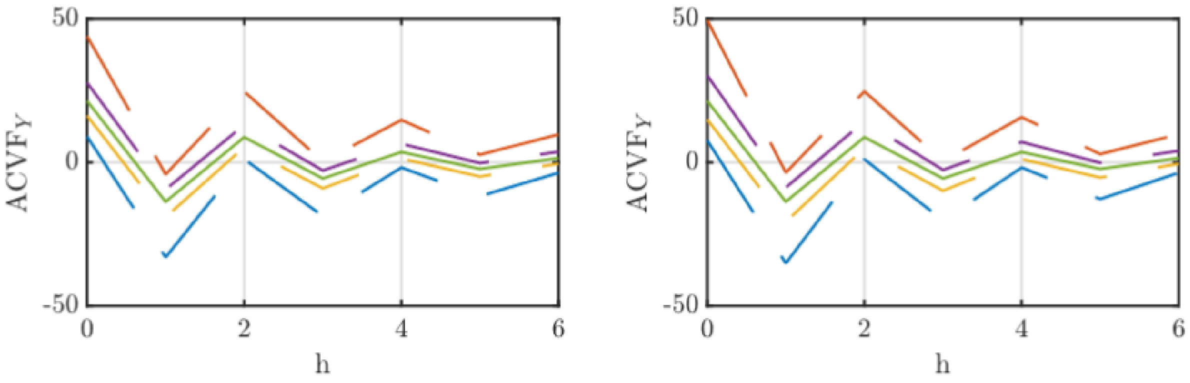


Fig. 2: Sample trajectories of the VAR(1) model components and their product for the Gaussian (left panels) and Student's t distribution (right panels). The parameters correspond to Case 1, i.e. $\phi_{11} = 0.8$, $\phi_{22} = -0.8$, $\phi_{12} = \phi_{21} = 0$ and the residual vectors $Z_i(t), i = 1, 2$ are independent with $\eta = 5$ for the Student's t distribution and $\sigma^2_{Z,1} = \sigma^2_{Z,2} = \frac{\eta}{\eta-2}$ for the Gaussian one.





Special cases

2. Correlated noise – $\mathrm{corr}(Z_1, Z_2) = \rho_Z$

ρ_Z influences the product distribution

$$\mathbb{E}(Y(t)) = \frac{\rho_Z \sigma_{Z,1} \sigma_{Z,2}}{(1 - \phi_{11}\phi_{22})}$$

$$\mathbb{V}\mathrm{ar}(Y(t)) = \frac{m_Z - \sigma_{Z,1}^2 \sigma_{Z,2}^2 - 2\rho_Z^2 \sigma_{Z,1}^2 \sigma_{Z,2}^2}{1 - (\phi_{11}\phi_{22})^2} + \frac{\sigma_{Z,1}^2 \sigma_{Z,2}^2}{(1 - \phi_{11}^2)(1 - \phi_{22}^2)} + \frac{\rho_Z^2 \sigma_{Z,1}^2 \sigma_{Z,2}^2}{(1 - \phi_{11}\phi_{22})^2}$$

$$\mathrm{ACVF}_Y(h) = (\phi_{11}\phi_{22})^h \left[\frac{m_Z - \sigma_{Z,1}^2 \sigma_{Z,2}^2 - 2\rho_Z^2 \sigma_{Z,1}^2 \sigma_{Z,2}^2}{1 - (\phi_{11}\phi_{22})^2} + \frac{\sigma_{Z,1}^2 \sigma_{Z,2}^2}{(1 - \phi_{11}^2)(1 - \phi_{22}^2)} + \frac{\rho_Z^2 \sigma_{Z,1}^2 \sigma_{Z,2}^2}{(1 - \phi_{11}\phi_{22})^2} \right]$$

= **0** for Gaussian distribution

$$m_Z = \mathbb{E}\left[Z_1^2(t) Z_2^2(t)\right]$$

Shape of the noise distribution is also important

$\rho_Z = 0.8$

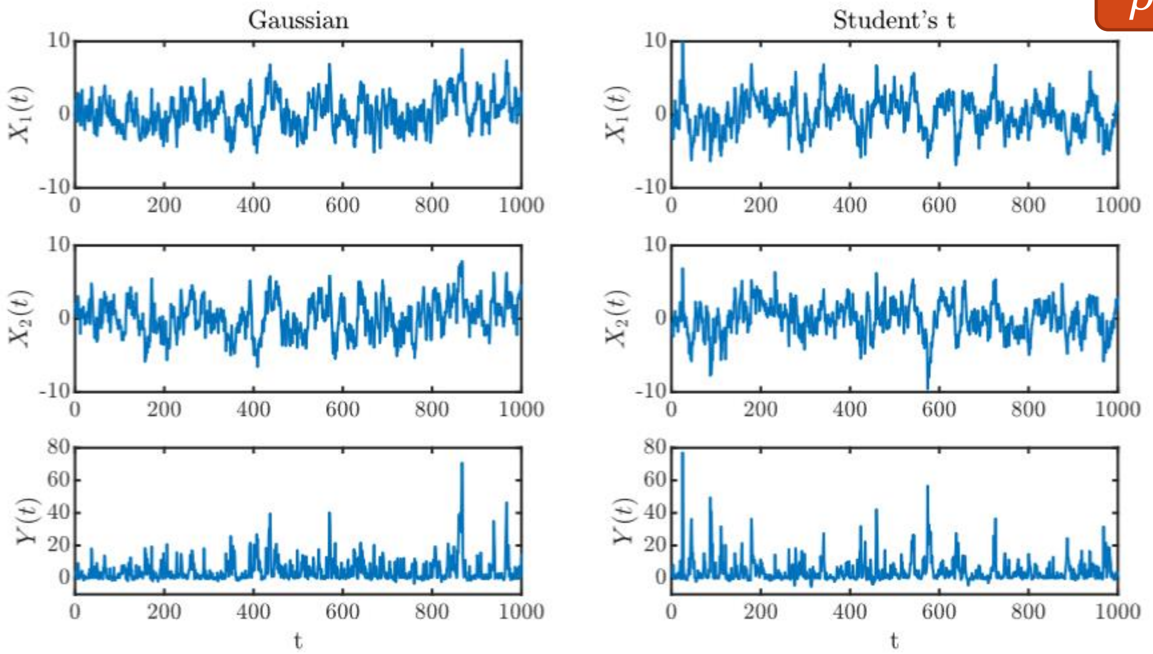


Fig. 4: Sample trajectories of the VAR(1) model components and their product for the Gaussian (left panels) and Student's t distribution (right panels). The parameters correspond to Case 2, i.e. $\phi_{11} = 0.8$, $\phi_{22} = 0.8$, $\phi_{12} = \phi_{21} = 0$ and the residual vectors $Z_i(t), i = 1, 2$ are correlated with $\rho_Z = 0.8$. $\eta = 5$ for the Student's t distribution and $\sigma^2_{Z,1} = \sigma^2_{Z,2} = \frac{\eta}{\eta-2}$ for the Gaussian one.

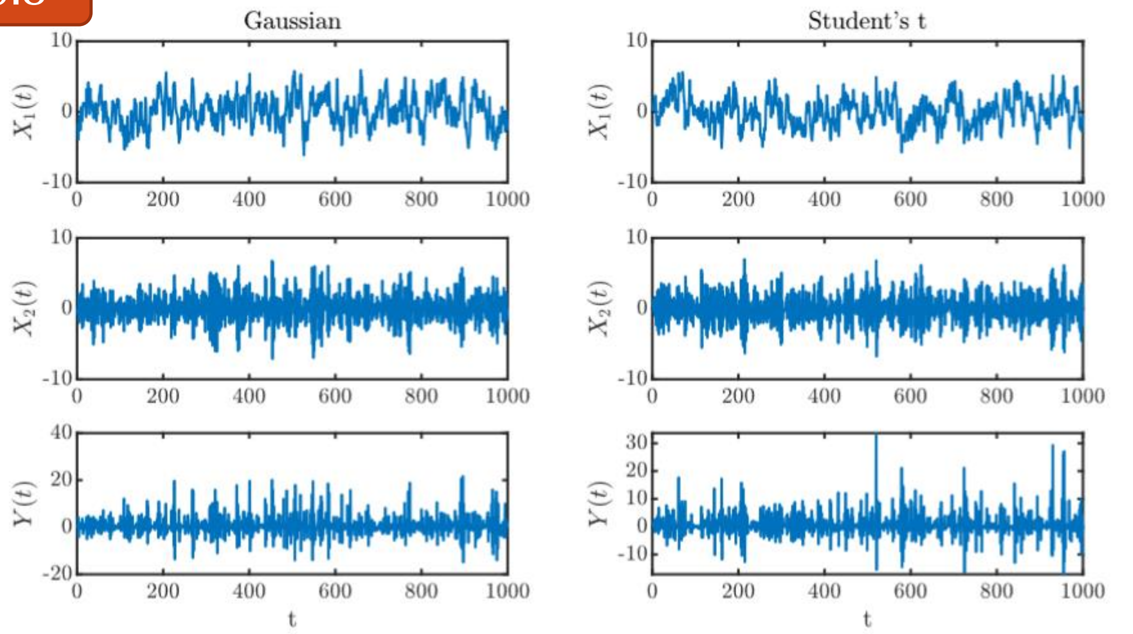
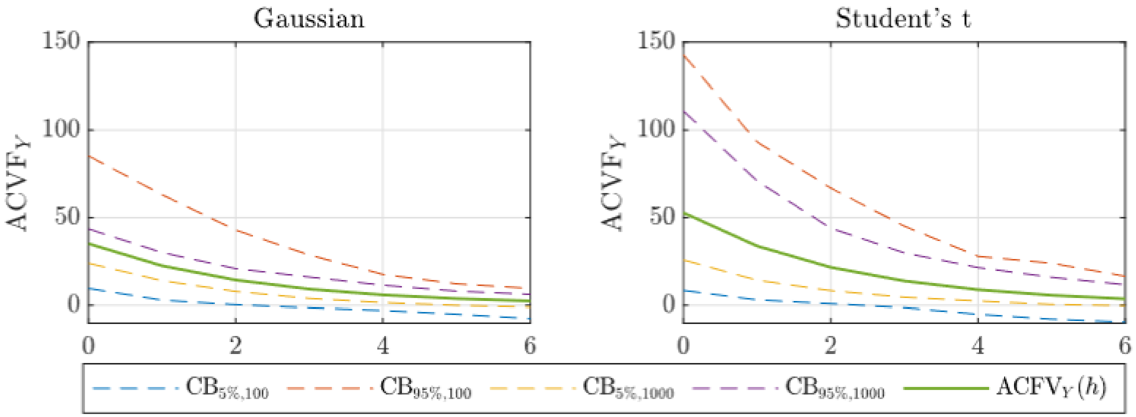
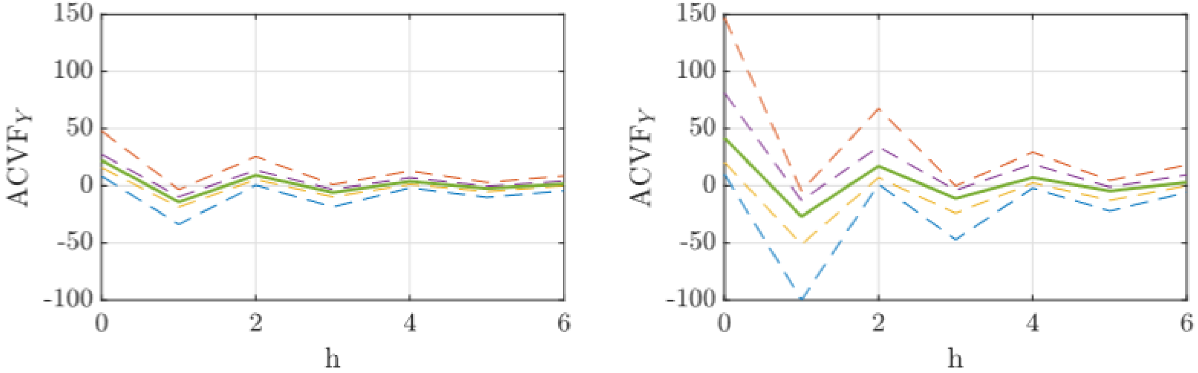


Fig. 5: Sample trajectories of the VAR(1) model components and their product for the Gaussian (left panels) and Student's t distribution (right panels). The parameters correspond to Case 2, i.e. $\phi_{11} = 0.8$, $\phi_{22} = -0.8$, $\phi_{12} = \phi_{21} = 0$ and the residual vectors $Z_i(t), i = 1, 2$ are correlated with $\rho_Z = 0.8$. $\eta = 5$ for the Student's t distribution and $\sigma^2_{Z,1} = \sigma^2_{Z,2} = \frac{\eta}{\eta-2}$ for the Gaussian one.





Special cases

3. Dependence through coefficients Φ

$$X_1(t) - \phi_{12} X_2(t-1) = Z_1(t) \qquad X_2(t) - \phi_{22} X_2(t-1) = Z_2(t).$$

$$\mathbb{E}(Y(t)) = \frac{\sigma^2_{Z,2}\phi_{12}\phi^2_{22}}{\phi_{22}(1-\phi^2_{22})}$$

Mean is a function of Φ and $\sigma^2_{Z,2}$

$$Var(Y(t)) = \phi^2_{12}\left[\frac{\phi^2_{22}(\kappa_Z - 3\sigma^4_{Z,2})}{1-\phi^4_{22}} + \frac{(1+\phi^2_{22})\sigma^4_{Z,2}}{(1-\phi^2_{22})^2}\right] + \frac{\sigma^2_{Z,1}\sigma^2_{Z,2}}{1-\phi^2_{22}},$$

$$\text{ACVF}_Y(h) = \phi^2_{12}\phi^{2h}_{22}\left[\frac{\phi^2_{22}(\kappa_Z - 3\sigma^4_{Z,2})}{1-\phi^4_{22}} + \frac{2\sigma^4_{Z,2}}{(1-\phi^2_{22})^2}\right]$$

Shape of the noise distribution is also important

$$\kappa_Z = \mathbb{E}\left[Z^4_2(t)\right]$$

= **0** for Gaussian distribution

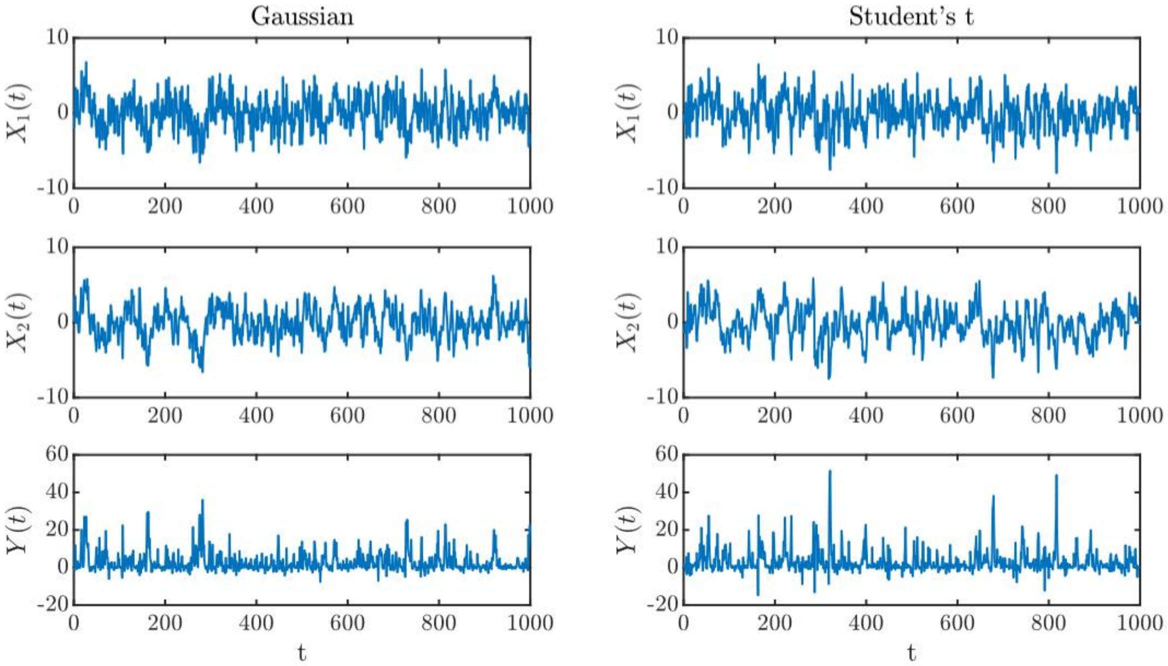
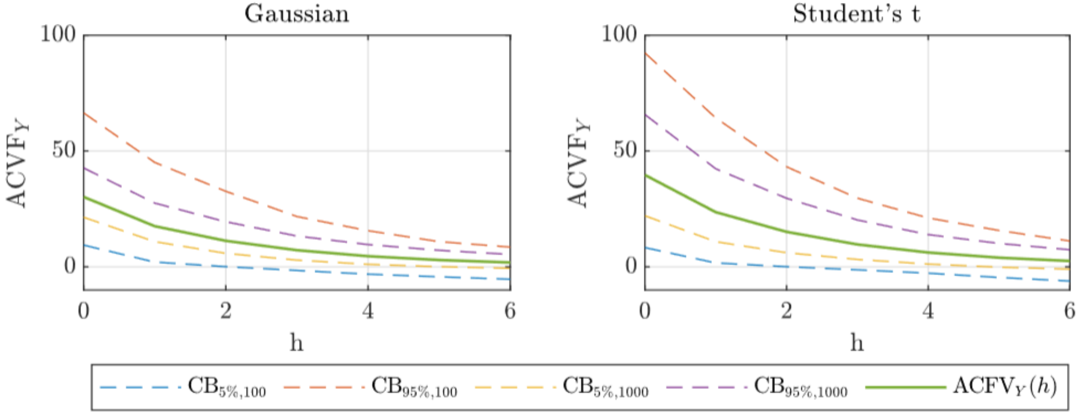
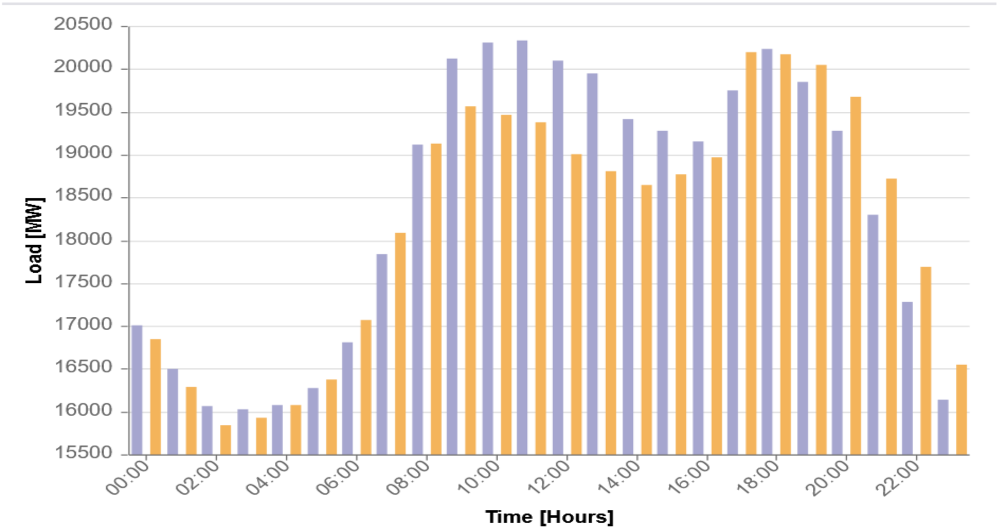


Fig. 8: Sample trajectories of the VAR(1) model components and their product for the Gaussian (left panels) and Student's t distribution (right panels). The parameters correspond to Case 3, i.e. $\phi_{12} = 0.8$, $\phi_{22} = 0.8$, $\phi_{11} = \phi_{21} = 0$ and the residual vectors $Z_i(t), i = 1, 2$ are independent with $\eta = 5$ for the Student's t distribution and $\sigma^2_{Z,1} = \sigma^2_{Z,2} = \frac{\eta}{\eta-2}$ for the Gaussian one.



Electricity market

- Electricity trade is done mostly on day-ahead markets
- Production/demand volumes are not known exactly on the day ahead delivery – only forecasts can be used
- TSO publishes day-ahead forecasts of load and generation
- The forecasts have errors
- Electricity trade needs to be balanced prior to delivery



Electricity market

- Forecasted volume < actual volume

  ➡ difference needs to be bought in the intraday (balancing) market

- Forecasted volume > actual volume

  ➡ difference needs to be sold in the intraday (balancing) market

- $$\textbf{Extra balancing cost}(\boldsymbol{t}) = \underbrace{\left[\boldsymbol{P}^{DA}(\boldsymbol{t}) - \boldsymbol{P}^{B}(\boldsymbol{t})\right]}_{\text{Price difference}} * \underbrace{\left[\widehat{\boldsymbol{V}}^{DA}(\boldsymbol{t}) - \boldsymbol{V}^{act}(\boldsymbol{t})\right]}_{\text{Volume imbalance}}$$

- It is positive (cost) if additional volume is bought at higher prices or it is sold at lower prices
- It is negative (profit), if additional volume is bought at lower prices or it is sold at higher prices

Polish electricity market

- Hourly day–ahead & balancing prices,
- Hourly TSO load forecast errors
- Each hour modelled separately

$$\frac{1}{\sigma_1}\left[P^{DA}(t) - P^{B}(t)\right] = X_1(t)$$

$$\frac{1}{\sigma_2}\left[\hat{V}^{DA}(t) - V^{act}(t) \ - \mu_2\right] = X_2(t)$$

Cost of balancing

$$W_1(t) = \sigma_1 X_1(t)[\sigma_2 X_2(t) + \mu_2]$$

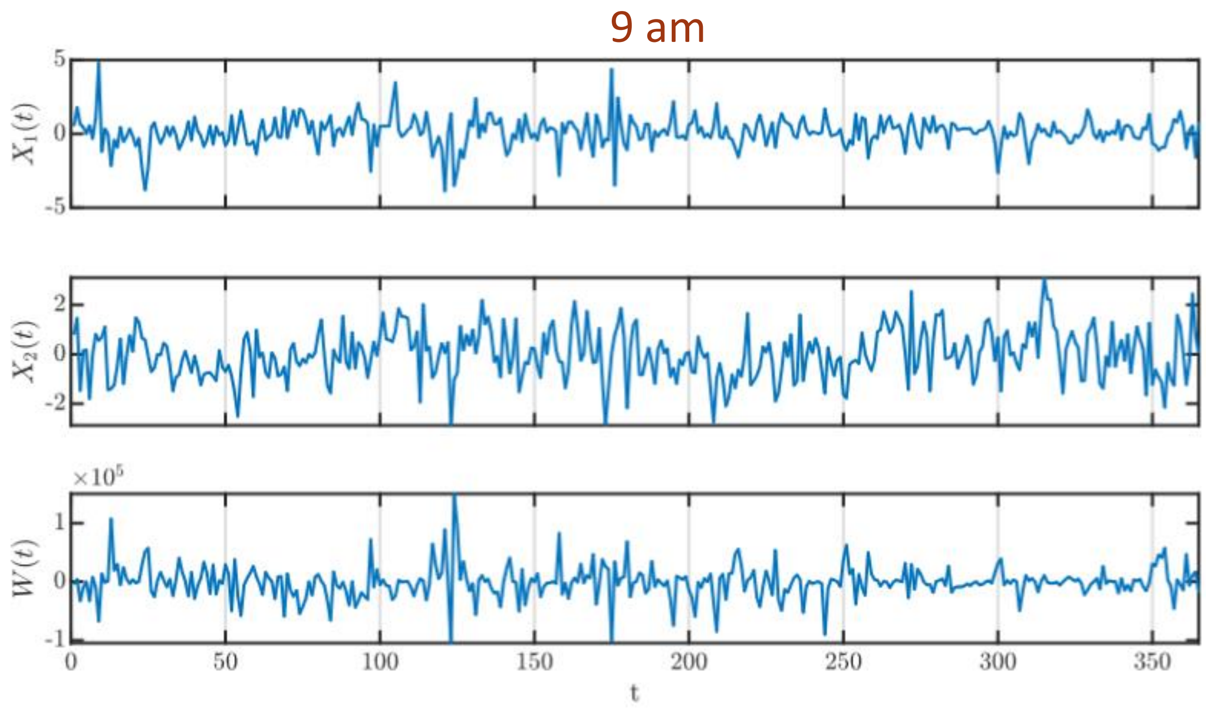


Fig. 10: The analyzed Polish electricity market data from the time period 1.1.2019-31.12.2019. Top panel: the hourly spread for 9am, divided by its standard deviation, i.e. $X_1(t)$. Middle panel: the hourly, standardized, TSO load forecast errors for 9am, i.e. $X_2(t)$. Bottom panel: the product of hourly spread and load forecast errors, $W(t) = \sigma_1 X_1(t)(\sigma_2 X_2(t) + \mu_2)$.

Fitted VAR(1) model

- Dependence through bivariate resiudals (Z_1, Z_2) distribution
  - significant correlation
- Autoregression in load forecast errors
- Cross-dependence coefficients $\Phi_{12}, \Phi_{2,1}$close to 0
- Good fit (according to the KS test) of the NIG distribution to model residuals Z_1, Z_2

	Model parameters					Test p-values		
						$\rho_Z = 0$	NIG	
Hour	Φ_{11}	Φ_{12}	Φ_{21}	Φ_{22}	ρ_Z		Z_1	Z_2
1	-0.02	-0.02	0.06	0.31	0.16	0.002	0.198	0.232
2	0.11	0.01	0.02	0.29	0.15	0.004	0.704	0.822
3	0.04	-0.01	0.10	0.29	0.18	0.000	0.989	0.642
4	0.08	0.05	0.10	0.28	0.15	0.004	0.704	0.580
5	0.07	0.08	0.11	0.41	0.12	0.023	0.917	0.704
6	0.07	-0.03	0.11	0.32	0.20	0.000	0.822	0.822
7	0.17	-0.06	0.13	0.22	0.23	0.000	0.169	0.519
8	0.23	0.06	-0.05	0.27	0.11	0.029	0.407	0.580
9	0.14	0.09	0.02	0.34	0.12	0.027	0.232	0.519
10	0.16	0.09	0.03	0.36	0.17	0.001	0.232	0.311
11	0.15	0.12	0.03	0.33	0.21	0.000	0.169	0.232
12	0.18	0.13	0.02	0.32	0.24	0.000	0.822	0.974
13	0.18	0.14	0.03	0.32	0.26	0.000	0.917	0.765
14	0.14	0.13	0.03	0.32	0.27	0.000	0.704	0.917
15	0.14	0.13	0.04	0.32	0.25	0.000	0.642	0.974
16	0.18	0.07	0.02	0.43	0.26	0.000	0.580	0.407
17	0.24	0.10	0.02	0.50	0.26	0.000	0.462	0.873
18	0.27	0.08	-0.01	0.49	0.16	0.002	0.198	0.974
19	0.23	0.04	-0.01	0.43	0.22	0.000	0.462	0.311
20	0.15	-0.01	0.05	0.49	0.22	0.000	0.357	0.765
21	0.13	0.03	0.04	0.30	0.27	0.000	0.704	0.084
22	0.12	0.00	0.07	0.33	0.29	0.000	0.198	0.873
23	0.08	0.07	0.15	0.34	0.23	0.000	0.822	0.031
24	0.10	0.11	0.04	0.35	0.20	0.000	0.822	0.822

Fitted VAR(1) model - residuals

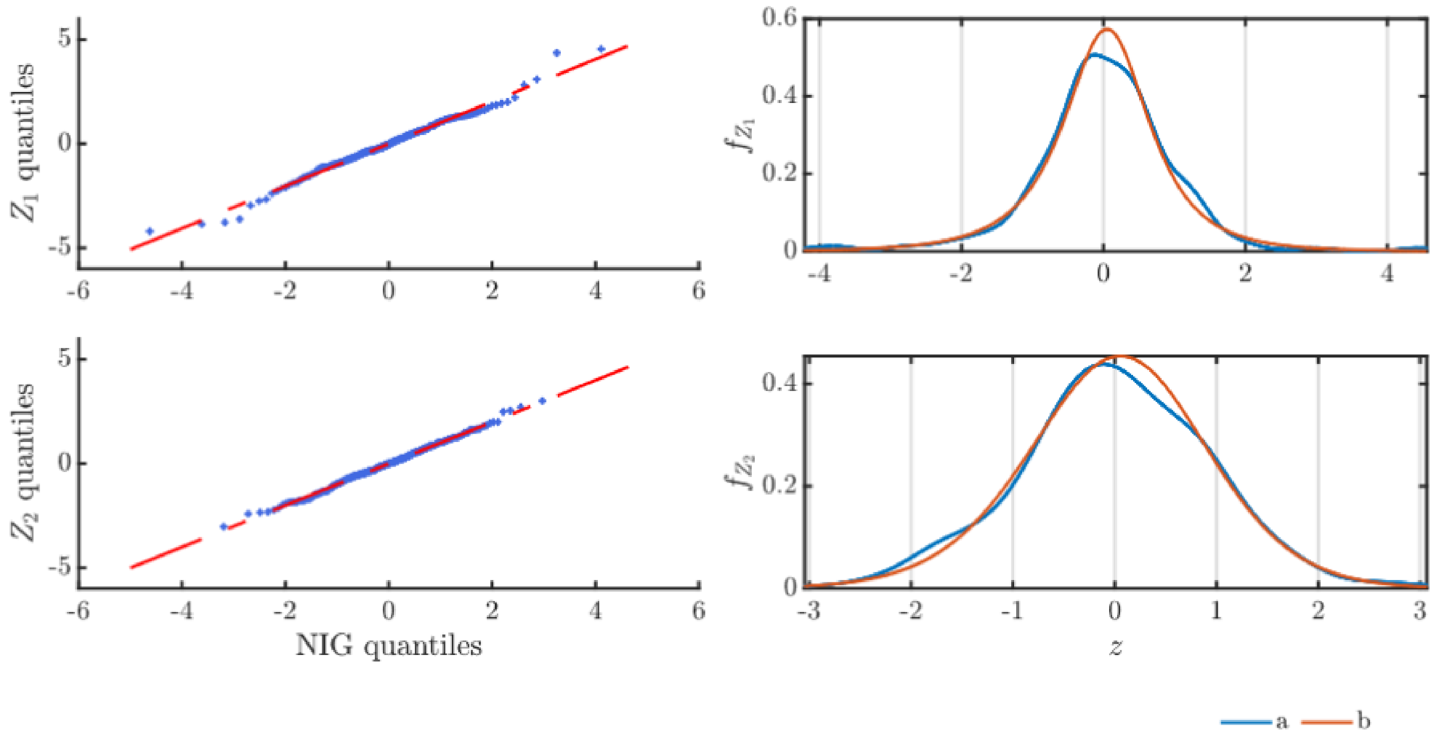
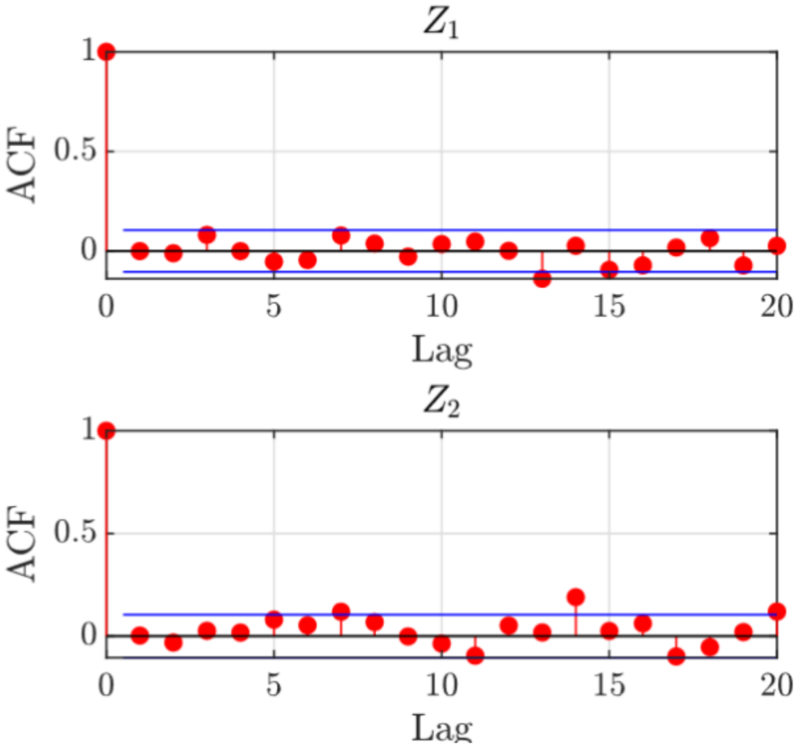


Fig. 14: The quantile-quantile plots for the NIG distribution (left panels) fitted to time series corresponding to $\{Z_1(t)\}$ (top panels) and $\{Z_2(t)\}$ (bottom panels). In the right panels the empirical PDFs (a, blue colour) corresponding to $\{Z_1(t)\}$, $f_{Z_1}(z)$, and $\{Z_2(t)\}$, $f_{Z_2}(z)$, together with the fitted PDFs for the NIG (b, red colour) distribution are plotted.



Product of the VAR(1) components

$$\begin{aligned}\mathrm{ACVF}_W(l) &= \sigma_1^2\sigma_2^2\Bigg\{\mathrm{ACVF}_{X_1X_2}(l) + \frac{\mu_2^2}{\sigma_2^2}\mathrm{ACVF}_{X_1}(l) \\ &\quad + \frac{\mu_2}{\sigma_2}\left[\mathrm{Cov}(X_1(t)X_2(t), X_1(t+l)) + \mathrm{Cov}(X_1(t), X_1(t+l)X_2(t+l))\right]\Bigg\} \\ &= \sigma_1^2\sigma_2^2\left[\mathrm{ACVF}_Y(l) + \frac{\mu_2^2}{\sigma_2^2}\frac{\sigma_{Z,1}^2\phi_{11}^l}{1-\phi_{11}^2} + \frac{\mu_2}{\sigma_2}\frac{\mathbb{E}(Z_1^2Z_2)\phi_{11}^l(1+\phi_{22}^l)}{1-\phi_{11}^2\phi_{22}}\right],\end{aligned}$$

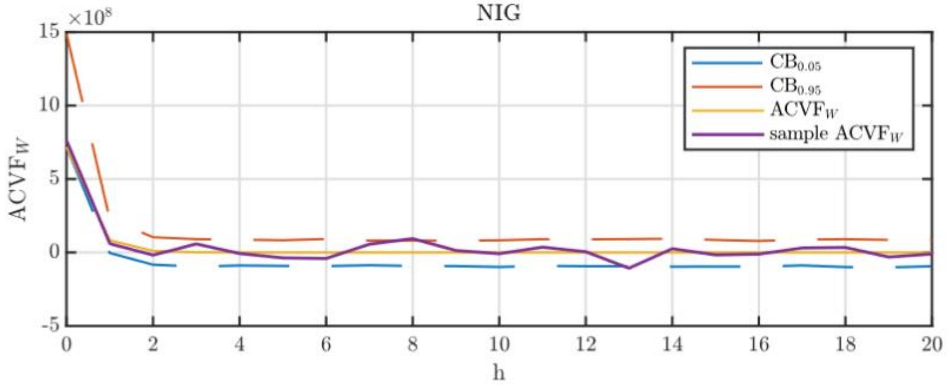
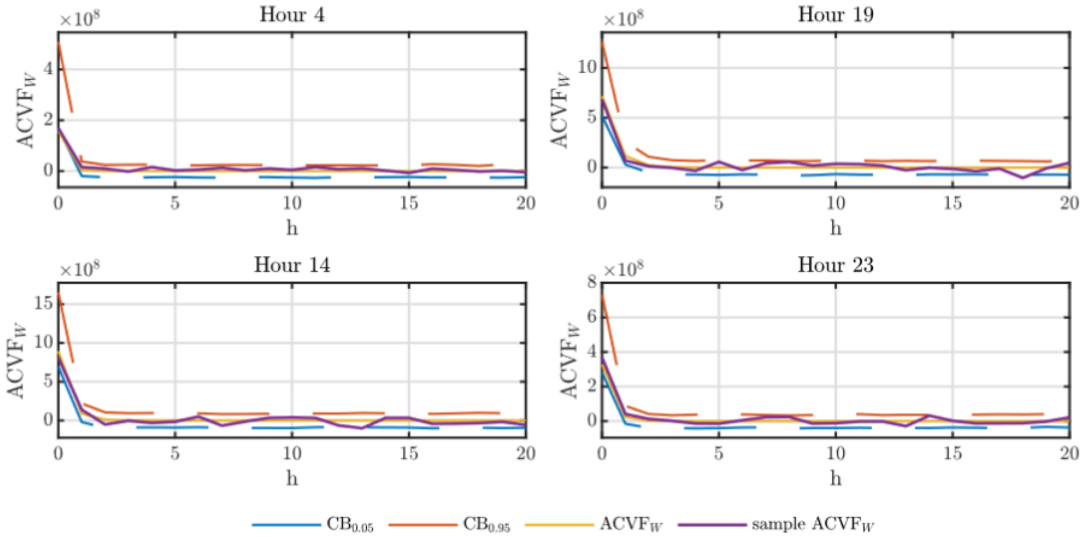
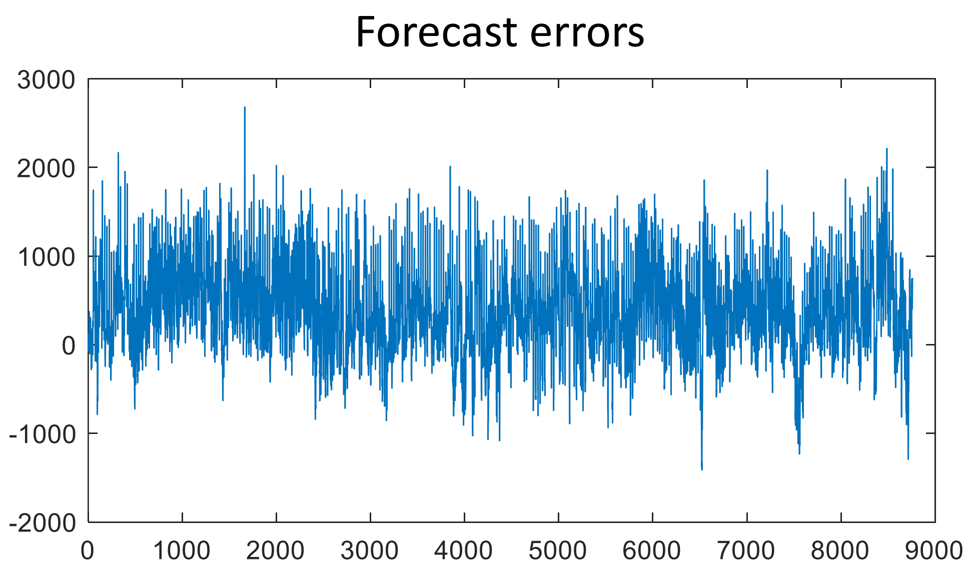


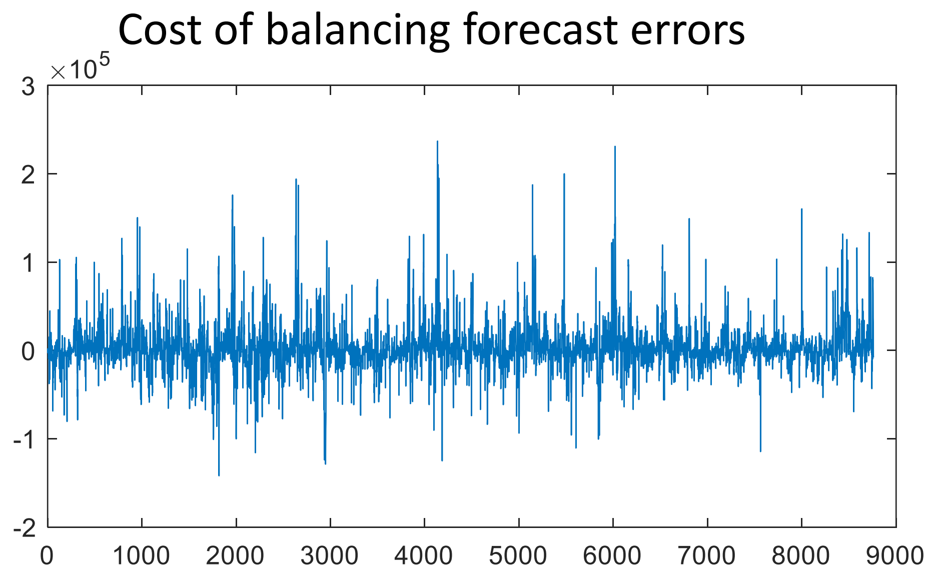
Fig. 15: The autocovariance function of the product of spread and load prediction errors for 9am, $\mathrm{ACVF}_W(l)$. The empirical autocovariance is plotted with a violet colour, while the theoretical values calculated for the fitted VAR(1) model, see Eq. (35), are plotted with a yellow colour. Additionally 5% and 95% CBs obtained from Monte Carlo simulations of the fitted model are plotted with dashed lines.



Implications

- Approach consistent for both variables and their product
  - can be used e.g. for risk management, like Value at Risk (VaR) calculation
- Cost of balancing is a direct economical cost of forecast errors
  - can be an alternative, economically grounded, measure for forecast evaluation





Thank you !