Consistent asset modelling with random coefficients and switches between regimes

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joint work with

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Outline

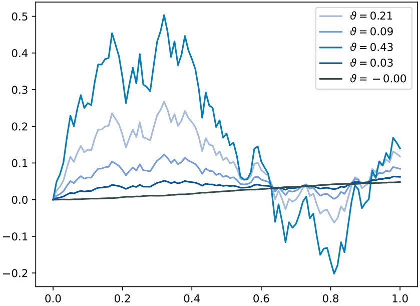
Introduction: Parameter Uncertainty with Randomisation

Let the assets prices be modelled in an exponential way with as $S(t) = S(0)e^{Y(t)}$ with $(Y(t))_t$ the asset's log-price process.
Let the coefficients of its SDE be functions of a random variable ϑ, called the *randomiser*, and let us then denote it by $Y^\vartheta(t)$:

$$\mathrm{d}Y^\vartheta(t) = b(t,\vartheta)\,\mathrm{d}t + \sigma(t,\vartheta)\,\mathrm{d}W(t), \quad Y^\vartheta(0) = 0$$

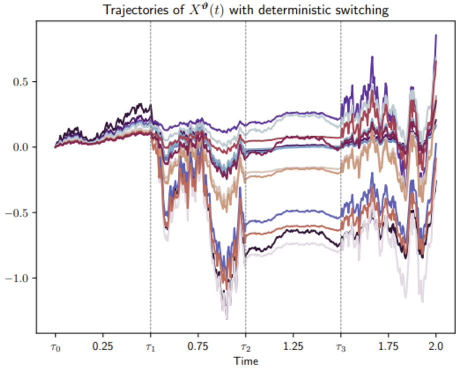
E.g.

$$\mathrm{d}Y^\vartheta(t) = \left(r - \frac{\vartheta^2}{2}\right)\mathrm{d}t + \vartheta\,\mathrm{d}W(t), \quad Y^\vartheta(0) = 0$$



Deterministic switching and Randomisation

$$\vartheta_0 \sim \mathcal{N}(.05, .1^2),\ \vartheta_1 \sim \mathcal{N}(.8, .5^2), \vartheta_2 \sim \mathcal{N}(.05, .1^2), \vartheta_1 \sim \mathcal{N}(.8, .5^2)$$



Deterministic switching and Randomisation

This process is an example of a process we call a composite process $X^{\vartheta}(t)$ with deterministic switches and randomised volatility coefficients,

$$\mathrm{d}X^{\vartheta}(t) = \sum_{j=0}^{3} \mathbb{1}_{t\in[\tau_j,\tau_{j+1})} \left(\left(r - \frac{\vartheta_j^2}{2} \right) \mathrm{d}t + \vartheta_j \,\mathrm{d}\widetilde{W}(t) \right), \tag{1}$$

with $r = 0.05$ the short interest rates,
for some driving Brownian motion $\widetilde{W}(t)$,
a random vector $\boldsymbol{\vartheta} = (\vartheta_0, \vartheta_1, \vartheta_2, \vartheta_3)$ measurable at the initial time.
and deterministic switches at $\tau_1 = 0.5, \tau_2 = 1, \tau_3 = 1.5$.

This process alternates between two types of regimes characterised by low and high volatility, expressed through randomisers ϑ_j, $0 \leq j \leq 3$.

Mixture models, Randomisation, local volatility models

- Mixture models are nicely interpretable, leading to good results for implied volatility for European type options.
- For path-dependent problems, one should take into account nested expectation issues, see Piterbarg (2003). E.g. the nested expectation problem of a compound put option with two exercise dates T_1 and T_2 at strikes K_1 and K_2.
- Brigo and Mercurio (2000) present a nice link with local volatility models.
- Local volatility models have proven their efficiency, and are in a complete market setting.
- In general, the local volatility functions are difficult to estimate.
- See also Grzelak (2022a,b) for an interesting study of randomisation with several applications and a deeper study of this link with local volatility models.

Main References

- L. A. Grzelak (2022a), On Randomization of Affine Diffusion Processes with Application to Pricing of Options on VIX and S&P 500. arXiv 2208.12518.
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- Wolf, F.L., Deelstra, G., Grzelak, L.A. (2024), Consistent asset modelling with random coefficients and switches between regimes, Mathematics and Computers in Simulation.

In this talk

- We study models which allow for uncertainty in the parametrisation of the stochastic dynamics and we take into account possible different behaviours across various times or regimes.
- We start by constructing a model with random parameters, where the switching between regimes can be dictated either deterministically or stochastically.
- We further present the equivalent modelling through local volatility models.
- We derive characteristic functions, providing a versatile tool with wide-ranging applications.
- Numerical section: option pricing. The impact of parameter uncertainty is analysed in a two-regime model, where the asset process switches between periods of high and low volatility.
- Jump dynamics are included in the paper but are here omitted for clarity.
- This talk is based upon:
  Wolf, F.L., Deelstra, G., Grzelak, L.A. (2024), Consistent asset modelling with random coefficients and switches between regimes, Mathematics and Computers in Simulation.

First model: Deterministic Switching

Let $b_j(t,z)$ and $\sigma_j(t,z)$ be real-valued functions which are finite for all $t \geq 0$ and bounded for all $z \in \mathbb{R}$.

Given a real-valued random variable ϑ_j, we introduce the *randomised component process* $Y_j^{\vartheta}(t)$, $t \geq 0$

$$Y_j^{\vartheta}(t) := \int_0^t b_j(u, \vartheta_j)\,\mathrm{d}u + \int_0^t \sigma_j(u, \vartheta_j)\,\mathrm{d}W_j(u). \tag{2}$$

For any realisation $\theta_j = \vartheta_j(\omega^*)$, we introduce the *conditional component process* $Y_j^{\theta}(t)$, $t \geq 0$ by

$$Y_j^{\theta}(t) := \int_0^t b_j(u, \theta_j)\,\mathrm{d}u + \int_0^t \sigma_j(u, \theta_j)\,\mathrm{d}W_j(u). \tag{3}$$

Composite process

Consider a random vector $\boldsymbol{\vartheta} = (\vartheta_0, \ldots, \vartheta_M)$ for some $M \in \mathbb{N}$ and the associated family of *randomised component processes* $\{Y_0^\vartheta, \ldots, Y_M^\vartheta\}$ with dynamics

$$dY_j^\vartheta(t) = b_j(t, \vartheta_j)\,\mathrm{d}t + \sigma_j(t, \vartheta_j)\,\mathrm{d}W_j(t), \quad Y_j^\vartheta(0) = 0, \tag{4}$$

with all Brownian motions $W_j(t)$ independent. Let $0 = \tau_0 < \tau_1 < \cdots < \tau_M$ be a sequence of *switching times* that define a *composite process*

$$X^{\boldsymbol{\vartheta}}(t) := x_0 + \sum_{j=0}^{M} Y_j^\vartheta(s_j(t)), \tag{5}$$

with time-shifts

$$s_j(t) := \begin{cases} 0, & t < \tau_j, \\ t - \tau_j, & \tau_j \leq t < \tau_{j+1}, \\ \tau_{j+1} - \tau_j, & \tau_{j+1} \leq t. \end{cases} \tag{6}$$

The conditional composite process

The SDE of the conditional composite process can be written as

$$\mathrm{d}X^{\boldsymbol{\theta}}(t) = \beta(t;\boldsymbol{\theta})\,\mathrm{d}t + \gamma(t;\boldsymbol{\theta})\,\mathrm{d}\widetilde{W}(t), \quad X^{\boldsymbol{\theta}}(0) = x_0, \tag{7}$$

with coefficients

$$\beta(t;\boldsymbol{\theta}) := \sum_{j=0}^{M} b_j(s_j(t),\theta_j)\mathbb{1}_{t\in[\tau_j,\tau_{j+1})}, \tag{8}$$

$$\gamma(t;\boldsymbol{\theta}) := \sum_{j=0}^{M} \sigma_j(s_j(t),\theta_j)\mathbb{1}_{t\in[\tau_j,\tau_{j+1})}. \tag{9}$$

The parameters $\boldsymbol{\theta} = \boldsymbol{\vartheta}(\omega) = (\vartheta_0(\omega),\ldots,\vartheta_M(\omega))$ are realisations of the randomisers.

Characteristic function of the processes

If the randomisers are independent, we observe the characteristic function of $X^{\boldsymbol{\vartheta}}(t)$

$$\varphi(u; X^{\boldsymbol{\vartheta}}(t)) = \mathrm{e}^{iux_0} \prod_{j=0}^{M} \varphi\left(u; Y_j^{\vartheta}(s_j(t))\right) \tag{10}$$

The characteristic function of $Y_j^{\vartheta}(t)$ is given by

$$\begin{aligned}\varphi(u; Y_j^{\vartheta}(t)) := \mathbb{E}\left[\exp\left(iuY_j^{\vartheta}(t)\right)\right] &= \int_{D_j} \varphi(u; Y_j^{\theta_j}(t))\, \mathrm{d}F_{\vartheta_j}(\theta_j) \\ &= \int_{D_j} \varphi(u; Y_j^{\theta_j}(t)) f_{\vartheta_j}(\theta_j)\, \mathrm{d}\theta_j,\end{aligned}$$

where D_j denotes the domain of the random variable ϑ_j, and F_{ϑ_j}, f_{ϑ_j} its cumulative distribution function and probability density function, respectively.

Discretisation of characteristic function

For $j \in \{0, \ldots, M\}$ let $N_j \in \mathbb{N}$ be the order of approximation. If the moments of the randomiser ϑ_j are finite for every $n_j \leq 2N_j$, $\mathbb{E}[\vartheta_j^{n_j}] < \infty$, then the characteristic function $\varphi(u; Y_j^{\vartheta}(t))$ is represented by the discretisation

$$\varphi(u; Y_j^{\vartheta}(t)) = \int_{D_j} f_{\vartheta_j}(\theta_j)\varphi(u; Y_j^{\theta_j}(t))\,\mathrm{d}\theta_j = \sum_{n_j=1}^{N_j} w_{n_j}\,\varphi(u; Y_j^{\theta_{n_j}}(t)) + \varepsilon_{N_j}(t,u).$$

Here, the weights/points $(w_{n_j}, \theta_{n_j})_{n_j=1}^{N_j}$ are the Gauss-quadrature pairs associated with integration against the weight function $f_{\vartheta_j}(\theta_j)$, and $\varepsilon_{N_j}(t,u)$ is the quadrature approximation error, which is bounded by

$$\varepsilon_{N_j}(t,u) \leq \sup_{\xi \in D_j} \frac{1}{(2N)!}\,\frac{\partial^{2N}}{\partial \theta^{2N}} \varphi(u; Y_j^{\theta_j}(t))\bigg|_{\theta=\xi}.$$

The computation of the weight/point pairs with the Golub-Welsch algorithm only requires knowledge about the moments of the randomisers, see e.g. Grzelak (2022a,b) or Grzelak et al. (2019).

Local volatility formulation of deterministic switching

We establish a local volatility model which behaves like the quadrature approximation of $X^{\boldsymbol{\vartheta}}(t)$: (Denote $\boldsymbol{\theta}_{|\boldsymbol{n}|} = (\theta_{n_0}, \ldots, \theta_{n_M})$)

$$\mathrm{d}\bar{X}(t) = \bar{\mu}\left(t, \bar{X}(t)\right) \mathrm{d}t + \bar{\sigma}\left(t, \bar{X}(t)\right) \mathrm{d}\bar{W}(t), \quad \bar{X}(0) = x_0, \tag{11}$$

$$\bar{\mu}(t,x) = \frac{\sum_{n_0,\ldots,n_M=1}^{N_0,\ldots,N_M} \left(\prod_{j=0}^{M} w_{n_j}\right) \beta(t; \boldsymbol{\theta}_{|\boldsymbol{n}|}) f\left(x; X^{\boldsymbol{\theta}_{|\boldsymbol{n}|}}(t)\right)}{\sum_{n_0,\ldots,n_M=1}^{N_0,\ldots,N_M} \left(\prod_{j=0}^{M} w_{n_j}\right) f\left(x; X^{\boldsymbol{\theta}_{|\boldsymbol{n}|}}(t)\right)}, \tag{12}$$

$$\bar{\sigma}^2(t,x) = \frac{\sum_{n_0,\ldots,n_M=1}^{N_0,\ldots,N_M} \left(\prod_{j=0}^{M} w_{n_j}\right) \gamma^2(t; \boldsymbol{\theta}_{|\boldsymbol{n}|}) f\left(x; X^{\boldsymbol{\theta}_{|\boldsymbol{n}|}}(t)\right)}{\sum_{n_0,\ldots,n_M=1}^{N_0,\ldots,N_M} \left(\prod_{j=0}^{M} w_{n_j}\right) f\left(x; X^{\boldsymbol{\theta}_{|\boldsymbol{n}|}}(t)\right)}. \tag{13}$$

where $f\left(x; X^{\boldsymbol{\theta}_{|\boldsymbol{n}|}}(t)\right)$ denotes the density of the process $X^{\boldsymbol{\theta}_{|\boldsymbol{n}|}}(t)$.
Then the SDE (11) has a unique, strong solution $\bar{X}(t)$ with probability density function

$$f(x; \bar{X}(t)) = \sum_{n_0,\ldots,n_M=1}^{N_0,\ldots,N_M} \left(\prod_{j=0}^{M} w_{n_j}\right) f\left(x; X^{\boldsymbol{\theta}_{|\boldsymbol{n}|}}(t)\right). \tag{14}$$

The proof of this result draws from an argument of identifying Fokker-Planck equations, see e.g. Brigo and Mercurio (2000) and Grzelak (2022b).

The characteristic function of $\bar{X}(t)$ is given for every $u \in \mathbb{R}$ and $t \geq 0$ by

$$\varphi(u; \bar{X}(t)) = e^{iux_0} \prod_{j=0}^{M} \sum_{n_j=1}^{N_j} w_{n_j} \varphi(u; Y_j^{\theta_{n_j}}(s_j(t))) \tag{15}$$

In summary, we have presented the local-volatility parametrisation under which its characteristic function (and its pdf) mimics that of the randomised composite process $X^{\boldsymbol{\vartheta}}(t)$.

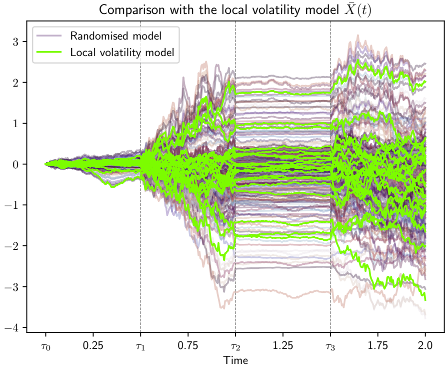

Figure: Sample paths of the randomised model with deterministic switching are contrasted with its associated local volatility model $\bar{X}(t)$.

Second model: Stochastic switching times

For some fixed number $M \in \mathbb{N}$, let $Y_j^{\vartheta}(t)$, $j \in \{0, \dots, M\}$ be randomised component processes and let $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_M)$ be the independent, stochastic sojourn times of these components. For every $j \geq 1$, we define the stochastic switching time by

$$\pi_j := \sum_{k=1}^{j} \zeta_k, \tag{16}$$

and set $\pi_0 := 0$. We further define the time shifts $s_j^{\boldsymbol{\zeta}}(t)$ by

$$s_j^{\boldsymbol{\zeta}}(t) := \begin{cases} 0, & t < \pi_j, \\ t - \pi_j, & t \in [\pi_j, \pi_{j+1}), \\ \zeta_{j+1}, & t \geq \pi_{j+1}, \end{cases} \tag{17}$$

for $j \in \{0, \dots, M-1\}$. For the final component Y_M^{ϑ}, the time shift is defined as $s_M^{\boldsymbol{\zeta}}(t) := (t - \pi_M)\mathbb{1}_{t \geq \pi_M}$. Then, the *randomised composite process with M stochastic switching times* is defined as

$$X^{\boldsymbol{\zeta}, \boldsymbol{\vartheta}}(t) := x_0 + \sum_{j=0}^{M} Y_j^{\vartheta}(s_j^{\boldsymbol{\zeta}}(t)). \tag{18}$$

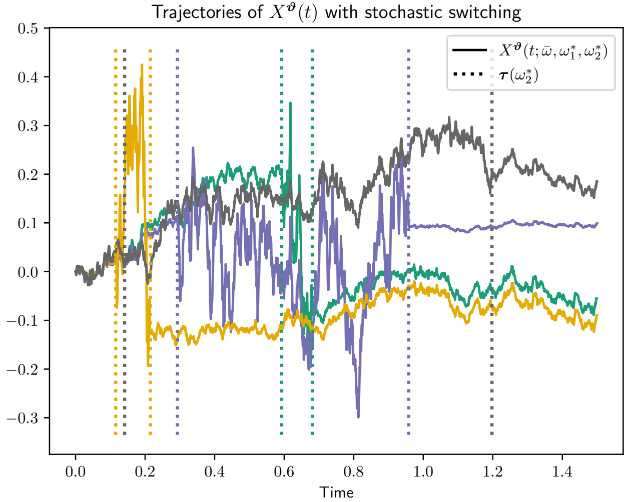

Figure: Sample paths of the stochastic switching model with two switches. Every trajectory uses the same underlying Brownian motion, all differences stem from the random samples of parameter randomisers and sojourn times.

Local volatility formulation of stochastic switching

Denote

$$\boldsymbol{\zeta}^M := (\zeta_1, \ldots, \zeta_M) \mid \sum_{\ell=1}^{M} \zeta_\ell < t. \tag{19}$$

and remark that

$$\begin{aligned} f(u; X^{\boldsymbol{\zeta},\boldsymbol{\vartheta}}(t)) &:= \int_{\mathbb{R}_+^M} f\left(u; X^{\boldsymbol{z},\boldsymbol{\vartheta}}(t)\right) \mathrm{d}F_{\boldsymbol{\zeta}^M}(\boldsymbol{z}) \\ &= \int_{\mathbb{R}_+^M} f_{\boldsymbol{\zeta}^M}(\boldsymbol{z}) f(u; X^{\boldsymbol{z},\boldsymbol{\vartheta}}(t)) \, \mathrm{d}\boldsymbol{z}, \end{aligned} \tag{20}$$

Denote the event $\mathcal{Z}_j(t) := \{\zeta_j < t - \sum_{\ell=0}^{j-1} \zeta_\ell\}$.
Remark that

$$f(x; X^{\boldsymbol{\zeta},\boldsymbol{\vartheta}}(t)) = \int_{\mathbb{R}_+^M} \prod_{\ell=1}^{M} f_{\zeta_\ell | \mathcal{Z}_\ell}(z_\ell) f(x; X^{\boldsymbol{z},\boldsymbol{\vartheta}}(t)) \, \mathrm{d}\boldsymbol{z}$$

Intuition about quadrature rule

Indeed, to factor the density, construct the successive conditioning

$$f_{\boldsymbol{\zeta}}^{M}(\boldsymbol{z}) = f_{\zeta_1|\{\zeta_1<t\}}(z_1) f_{\zeta_2|\{\zeta_2<t-\zeta_1\}}(z_2) \cdots f_{\zeta_M|\{\zeta_M<t-\sum_{j<M}\zeta_j\}}(z_M). \quad (21)$$

The first quadrature pair $(v_{\ell_1}, z_{\ell_1}), \ell_1 = 1, \ldots, L_1$ can be computed for $\zeta_1|\{\zeta_1 < t\}$ which is a right-truncated distribution.

For each point z_{ℓ_1}, we can compute dependent quadrature nodes $(v_{\ell_2}^{\ell_1}, z_{\ell_2}^{\ell_1}), \ell_2 = 1, \ldots, L_2$ from $\zeta_2|\{z_{\ell_1} + \zeta_2 < t\}$.

Eventually, we reach all pairs $(v_{\ell_j}^{\ell_k, k<j}, z_{\ell_j}^{\ell_k, k<j})$, $j = 1, \ldots M$.

$$
\begin{aligned}
f(x; X^{\boldsymbol{\zeta},\boldsymbol{\vartheta}}(t)) &= \int_{\mathbb{R}_+^M} \prod_{\ell=1}^{M} f_{\zeta_\ell | \mathcal{Z}_\ell}(z_\ell) f(x; X^{\boldsymbol{z},\boldsymbol{\vartheta}}(t)) \, \mathrm{d}\boldsymbol{z} \\
&\approx \sum_{\ell_1,\dots,\ell_M=1}^{L_1,\dots,L_M} \left(v_{\ell_1} v_{\ell_2}^{\ell_1} \cdots v_{\ell_M}^{\ell_1,\dots,\ell_{M-1}} \right) f\left(x; X^{(z_{\ell_1}, z_{\ell_2}^{\ell_1}, \dots, z_{\ell_M}^{\ell_1,\dots,\ell_{M-1}}),\boldsymbol{\vartheta}}(t) \right) \\
&=: \sum_{|\ell|=\mathbf{1}}^{|\boldsymbol{L}|} V_{|\boldsymbol{\ell}|} f(x; X^{\boldsymbol{z}_{|\boldsymbol{\ell}|},\boldsymbol{\vartheta}}(t)),
\end{aligned}
$$

with quadrature weights $V_{|\boldsymbol{\ell}|} = \prod_{\ell_j \in |\ell|} v_{\ell_j}^{\ell_1,\dots,\ell_j-1}$.
Each density $f(x; X^{\boldsymbol{z}_{|\boldsymbol{\ell}|},\boldsymbol{\vartheta}}(t))$ is the density of a randomised composite process with M deterministic switches given by
$\boldsymbol{\tau}_{|\boldsymbol{\ell}|} := (z_{\ell_1}, z_{\ell_1} + z_{\ell_2}^{\ell_1}, \dots, z_{\ell_1} + \dots + z_{\ell_M}^{\ell_1,\dots,\ell_{M-1}})$. For every composite process with deterministic switches, applying the quadrature approximation yields

$$
f(x; X^{\boldsymbol{\zeta},\boldsymbol{\vartheta}}(t)) \approx \sum_{|\ell|=\mathbf{1}}^{|\boldsymbol{L}|} V_{|\boldsymbol{\ell}|} \sum_{n_0,\dots,n_M=1}^{N_0,\dots,N_M} \left(\prod_{j=0}^{M} w_{n_j} \right) f(x; X^{\boldsymbol{z}_{|\boldsymbol{\ell}|},\boldsymbol{\theta}_{|\boldsymbol{n}|}}(t)).
$$

Local volatility and extension stochastic switching

- The quadrature discretisation is thus obtained for stochastic switching (with $M \in \mathbb{N}$).
- A local volatility formulation $\widehat{X}(t)$ is obtained, namely with the solution of this local-volatility type SDE exhibiting marginal distributions that align with those of the quadrature discretisation of the randomized composite process.
- This can be extended to the case of fully stochastic switching, with the number of regimes at t modeled by a random variable. We denote the related fully stochastic switching local volatility formulation by $\tilde{X}(t)$.

Numerical Results

- Implied volatility (IV) surfaces obtained for European call options.
- We model the asset with the local volatility models corresponding to deterministic switching, $\bar{X}(t)$, stochastic switching, $\widehat{X}(t)$, and fully stochastic switching $\widetilde{X}(t)$.
- We compute all option prices needed for implied volatility computations with the COS-method of Fang and Oosterlee (2009), based on the obtained characteristic functions.

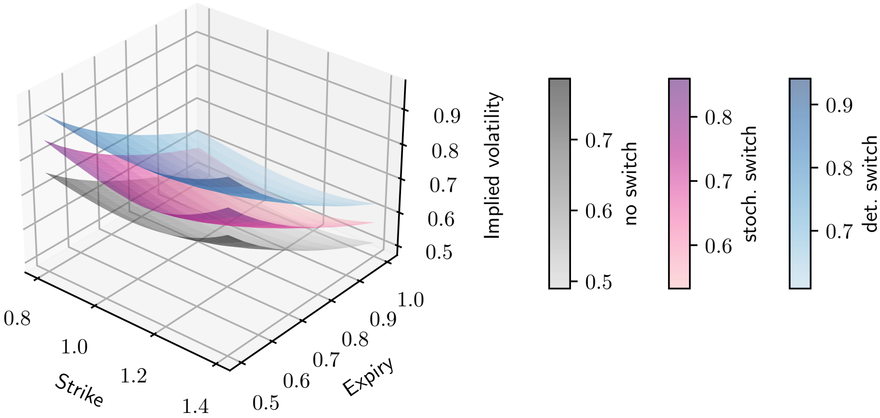

Figure: Implied volatility surface as the expiry T of the option decreases. The underlying asset is modelled with one switch at time $T/2$ in the deterministic switch model $\bar{X}(t)$, and one exponentially distributed switch such that $\mathbb{E}[\zeta_1] = T/2$ in the stochastic switching model $\widehat{X}(t)$, both times the composite process switches from a regime with 'calm' randomiser $\vartheta_0 \sim \mathcal{N}(0.15, 0.1^2)$ to one with an 'excited' randomiser $\vartheta_1 \sim \mathcal{N}(0.3, 1)$. We also consider the randomised model with no switch, where the randomiser is that of the 'excited' randomizer regime throughout.

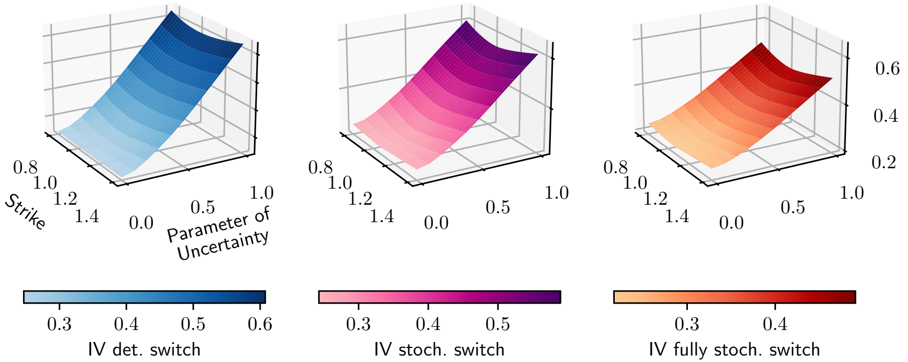

Figure: We fix $T = 1$ and we consider the two-regime example with a 'calm' and an 'excited' randomiser distribution, resp. $\mathcal{N}(\nu_0, \xi_0^2)$ and $\mathcal{N}(\nu_1, \xi_1^2)$. The implied volatility surfaces are obtained for a range of values for the 'excited' randomiser's standard deviation ξ_1. Considered are the deterministic and stochastic switching models with one switch each, as well as the fully stochastic switching model with a random number of switches.

Conclusions and further extensions

- All experiments show that randomisation and switching have an important impact on the model's implied volatility.
- In another approach, we drive the switching between randomised component processes with a Markov chain, resulting in a Markov-modulated randomised model, also called regime-switching randomised model.
- Characteristic function is obtained in all models, enabling many applications.

Thank you for your attention!