The multidimensional COS method for option pricing

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- Aim: solve numerically

$$\int_{\mathbb{R}^d} w(\boldsymbol{x})g(\boldsymbol{x})d\boldsymbol{x}.$$

- g density, e.g., of log-returns
- w function of interest
- solutions: Fourier-methods (COS, Fang&Oosterlee (2009), Lewis, Wavelets,...), MC, numerical integration.
- Example for w indicator function or "digital put option"

$$w(\boldsymbol{x}) = \prod_{h=1}^{d} 1_{[0,K_h]}(e^{x_h})$$

or basket option

$$w(\boldsymbol{x}) = \max(K - \sum_{h=1}^{d} e^{x_h}, 0).$$

COS I Love You (With Apologies to Slade) - M. Staunton

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$$\widehat{g}(\boldsymbol{u}) := \int_{\mathbb{R}^d} g(\boldsymbol{x}) e^{i\boldsymbol{u}\cdot\boldsymbol{x}} d\boldsymbol{x}.$$

- Sum of independent random variables, Lévy processes, Heston model,...

- $w \notin \mathcal{L}_1$ but $\widehat{w}$ is given, e.g., for basket option:

$$\widehat{w}(\boldsymbol{z}) = K^{(1+i\sum_{h=1}^{d} z_h)} \frac{\prod_{h=1}^{d} \Gamma(iz_h)}{\Gamma\left(i\sum_{h=1}^{d} z_h + 2\right)}, \quad \Im\{z_h\} < 0.$$

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Damping

- For scaling factor $\lambda > 0$, shift parameter $\boldsymbol{\mu} \in \mathbb{R}^d$ and a damping factor $\boldsymbol{\alpha} \in \mathbb{R}^d$, define the *damped density* by

$$f(\boldsymbol{x}) = \lambda e^{\boldsymbol{\alpha}\cdot(\boldsymbol{x}+\boldsymbol{\mu})} g(\boldsymbol{x}+\boldsymbol{\mu}), \quad \boldsymbol{x} \in \mathbb{R}^d$$

  and the *damped payoff function* by

$$v(\boldsymbol{x}) = \frac{1}{\lambda} e^{-\boldsymbol{\alpha}\cdot(\boldsymbol{x}+\boldsymbol{\mu})} w(\boldsymbol{x}+\boldsymbol{\mu}), \quad \boldsymbol{x} \in \mathbb{R}^d.$$

  By definition, it follows that

$$\int_{\mathbb{R}^d} w(\boldsymbol{x}) g(\boldsymbol{x}) d\boldsymbol{x} = \int_{\mathbb{R}^d} v(\boldsymbol{x}) f(\boldsymbol{x}) d\boldsymbol{x}.$$

- Further,

$$v \in \mathcal{L}_1,$$

$\widehat{f}$ is known if $\widehat{g}$ is known,

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Problem: f unknown

$$f \approx f 1_{[-\boldsymbol{L},\boldsymbol{L}]} \approx \sum_{\boldsymbol{0}\le\boldsymbol{k}\le\boldsymbol{N}}{}' a_{\boldsymbol{k}} e_{\boldsymbol{k}} 1_{[-\boldsymbol{L},\boldsymbol{L}]} \approx \sum_{\boldsymbol{0}\le\boldsymbol{k}\le\boldsymbol{N}}{}' c_{\boldsymbol{k}} e_{\boldsymbol{k}} 1_{[-\boldsymbol{L},\boldsymbol{L}]},$$

where

$$e_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{h=1}^{d} \cos\left(k_h \pi \frac{x_h + L_h}{2L_h}\right), \quad \boldsymbol{k} \in \mathbb{N}_0^d$$

and

$$\begin{aligned} a_{\boldsymbol{k}} &= \frac{1}{\prod_{h=1}^{d} L_h} \int_{[-\boldsymbol{L},\boldsymbol{L}]} f(\boldsymbol{x}) e_{\boldsymbol{k}}(\boldsymbol{x}) d\boldsymbol{x}\} \\ &\approx \frac{1}{\prod_{h=1}^{d} L_h} \int_{\mathbb{R}^d} f(\boldsymbol{x}) e_{\boldsymbol{k}}(\boldsymbol{x}) d\boldsymbol{x} \\ &= \frac{1}{2^{d-1} \prod_{h=1}^{d} L_h} \sum_{\boldsymbol{s}=(1,\pm1,\dots,\pm1)\in\mathbb{R}^d} \Re\left\{\widehat{f}\left(\frac{\pi}{2}\frac{\boldsymbol{s}\boldsymbol{k}}{\boldsymbol{L}}\right) \exp\left(i\frac{\pi}{2}\boldsymbol{s}\cdot\boldsymbol{k}\right)\right\} \\ &=: c_{\boldsymbol{k}}. \end{aligned}$$

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Definition

A function $f \in \mathcal{L}^1$ is called *COS-admissible*, if $\min_{h=1,\dots,d} L_h \to \infty$ implies

$$B(\boldsymbol{L}) := \sum_{\boldsymbol{k}\in\mathbb{N}_0^d}{}' \frac{1}{\prod_{h=1}^d L_h} \left| \int_{\mathbb{R}^d \setminus [-\boldsymbol{L},\boldsymbol{L}]} f(\boldsymbol{x}) e_{\boldsymbol{k}}(\boldsymbol{x}) d\boldsymbol{x} \right|^2 \to 0.$$

Proposition

Assume $f \in \mathcal{L}^1 \cap \mathcal{L}^2$ with $\int_{\mathbb{R}^d} |\boldsymbol{x}|^{2d} |f(\boldsymbol{x})|^2 d\boldsymbol{x} < \infty$. Then

$$B(\boldsymbol{L}) \le \frac{\frac{\pi^2}{3} \sum_{h=1}^d \left(\frac{\pi^2}{3}+1\right)^{h-1}}{d \min\limits_{h=1,\dots,d} L_h^{2d}} \int_{\mathbb{R}^d \setminus [-\boldsymbol{L},\boldsymbol{L}]} |\boldsymbol{x}|^{2d} |f(\boldsymbol{x})|^2 d\boldsymbol{x} +$$

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Junike & Stier (2024).

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Example

Stable and Pareto densities, bounded densities with existing moments.

Theorem

Assume that $f \in \mathcal{L}^1 \cap \mathcal{L}^2$ and that f is COS-admissible. For any $\varepsilon > 0$ there is $\boldsymbol{L} \in \mathbb{R}_+^d$ and $\boldsymbol{N} \in \mathbb{N}^d$ such that

$$\left\| f - {\sum_{\boldsymbol{0} \leq \boldsymbol{k} \leq \boldsymbol{N}}}' c_{\boldsymbol{k}} e_{\boldsymbol{k}} 1_{[-\boldsymbol{L}, \boldsymbol{L}]} \right\|_2 < \varepsilon.$$

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Solve the integral

- Let $\boldsymbol{M} = (M_1, ..., M_d) \in \mathbb{R}^d_+$ such that $\boldsymbol{M} \leq \boldsymbol{L}$. Then, intuitively,

$$\int_{\mathbb{R}^d} v(\boldsymbol{x}) f(\boldsymbol{x}) d\boldsymbol{x} \approx \int_{[-\boldsymbol{M},\boldsymbol{M}]} v(\boldsymbol{x}) \sum_{\boldsymbol{0} \leq \boldsymbol{k} \leq \boldsymbol{N}}{}' c_{\boldsymbol{k}} e_{\boldsymbol{k}}(\boldsymbol{x}) d\boldsymbol{x}$$

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  the last approximation works if $v \in \mathcal{L}_1$ and $\boldsymbol{M}$ large enough.

- *Classical COS method*: $\alpha = \boldsymbol{0}$ and $v_{\boldsymbol{k}}$ can be obtained explicitly or is approximated by numerical integration.

- *Damped COS method*: $\alpha \neq \boldsymbol{0}$ and obtain $\tilde{v}_{\boldsymbol{k}}$ through $\widehat{v}$.

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Theorem

Let $f \in \mathcal{L}^1 \cap \mathcal{L}^2$ *with* $f \geq 0$ *and* $\int_{\mathbb{R}^d} |\boldsymbol{x}|^{2d} |f(\boldsymbol{x})|^2 d\boldsymbol{x} < \infty$. *Let* $v : \mathbb{R}^d \to \mathbb{R}$ *with* $\|v\|_\infty \in (0, \infty)$. *Let* $n \geq 2$ *be even. Let*

$$m_h(n) := \int_{\mathbb{R}^d} x_h^n f(\boldsymbol{x}) d\boldsymbol{x} = i^{-n} \frac{\partial^n}{\partial u_h^n} \widehat{f}(\boldsymbol{u}) \bigg|_{\boldsymbol{u}=\boldsymbol{0}} \in (0, \infty), \quad h = 1, ..., d.$$

Assume that f *decays exponentially. Let* $\varepsilon > 0$ *be small enough. Define*

$$M_h := \left(\frac{3d \|v\|_\infty}{\varepsilon} m_h(n) \right)^{\frac{1}{n}}, \quad h = 1, ..., d,$$

and $\boldsymbol{L} = \boldsymbol{M} = (M_1, ..., M_d) \in \mathbb{R}_+^d$. *There is a* $\boldsymbol{N} \in \mathbb{N}_0^d$ *so that*

$$\left| \int_{\mathbb{R}^d} v(\boldsymbol{x}) f(\boldsymbol{x}) d\boldsymbol{x} - \sum_{\boldsymbol{0} \leq \boldsymbol{k} \leq \boldsymbol{N}}{}' c_{\boldsymbol{k}} v_{\boldsymbol{k}} \right| \leq \varepsilon.$$

Corollary

If, additionally, $v \in \mathcal{L}^1 \cap \mathcal{L}^2$ *with* $\int_{\mathbb{R}^d} |\boldsymbol{x}|^{2d} |v(\boldsymbol{x})|^2 d\boldsymbol{x} < \infty$ *and* v *decays exponentially, then*

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Corollary

The number of terms can be chosen by any $\boldsymbol{N} \in \mathbb{N}_0^d$ *such that*

$$\left| (2\pi)^{-d} \int_{\mathbb{R}^d} |\widehat{f}(\boldsymbol{u})|^2 d\boldsymbol{u} - \prod_{h=1}^{d} L_h {\sum_{\boldsymbol{0} \le \boldsymbol{k} \le \boldsymbol{N}}}' |c_{\boldsymbol{k}}|^2 \right| \le \frac{\varepsilon^2}{162 \|v\|_2^2}.$$

Junike & Stier (2024).

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Theorem

Assume $d = 1$. Assume all technical assumptions from the previous theorem are met. Assume $f \in C_b^{J+1}(\mathbb{R})$. For $k \in \{1, ..., J\}$ let

$$N \geq \left(12 \frac{2^{k+\frac{5}{2}} \left\| f^{(k+1)} \right\|_\infty L^{k+4} \left\| v \right\|_\infty}{k\pi^{k+1}\varepsilon} \right)^{\frac{1}{k}}.$$

Then,

$$\left| \int_{\mathbb{R}} v(\boldsymbol{x}) f(\boldsymbol{x}) d\boldsymbol{x} - {\sum_{0 \leq k \leq N}}' c_k \tilde{v}_k \right| \leq \varepsilon.$$

Junike (2024).

Number of terms

	d	Model	N	CPU time COS (num. int.)
Minimal N	1	BS	25	0.02
Cor.	1	BS	28	0.03
Junike (2024)	1	BS	34	0.03
Minimal $\boldsymbol{N}$	2	BS	65	5.35
Cor.	2	BS	116	17.02
Minimal $\boldsymbol{N}$	2	VG	55	8.55
Cor.	2	VG	154	68.20 (1092.2)

Table: Basket option $K = 200$, $\varepsilon = 10^{-3}$, $T = 1$ and $r = 0$. CPU time in milliseconds.

Theorem: Order of Convergence

- Let
$$\mathrm{err}(L(N),N) = \left| \int_{\mathbb{R}} v(x)f(x)dx - {\sum_{k=0}^{N}}' c_k v_k \right|.$$
- If f has semi-heavy tails, i.e., $|f(x)| \leq O(e^{-C_2|x|})$ and f is smooth then $\mathrm{err}(L(N),N)$ converges exponentially.
- If f has heavy tails with Pareto index $\theta > 0$, i.e., $|f(x)| \leq O(|x|^{-1-\theta})$ and f is smooth then
$$\mathrm{err}(L(N),N) \leq O\left(N^{-\theta}\right), \quad \text{as } N \to \infty.$$
- There is a multidimensional version without the assumption that f is differentiable to treat, e.g., the VG model.

Junike(2024) and Junike & Stier (2024).

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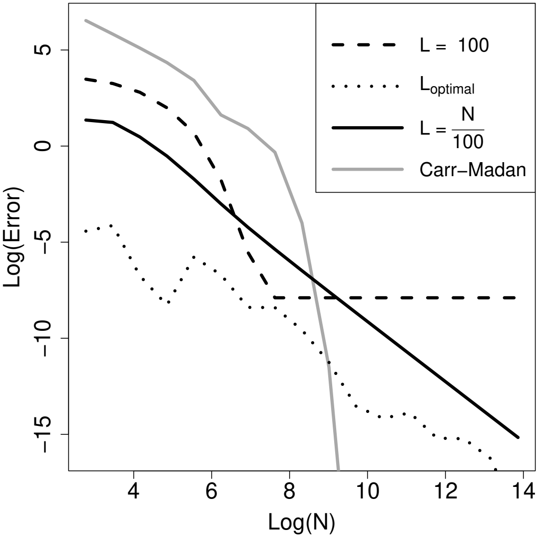
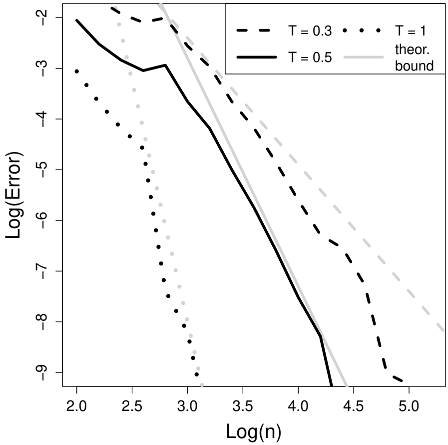
FMLS



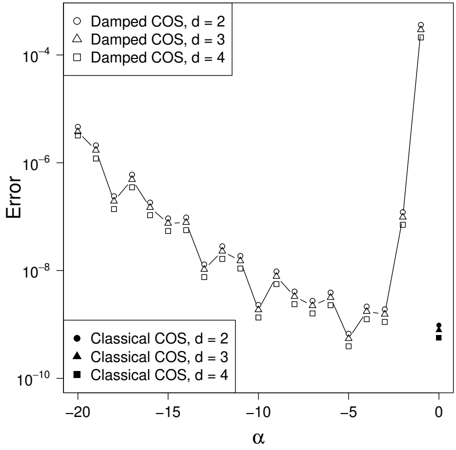
L = 100
$L_{optimal}$
$L = \frac{N}{100}$
Carr–Madan
Log(Error)
5
0
−5
−10
−15
4
6
8
10
12
14
Log(N)

VG



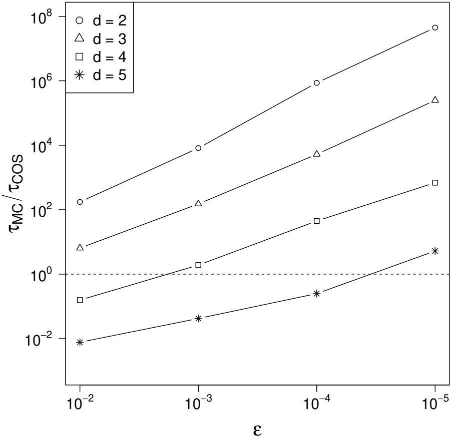
Error of the price for the VG model over the number of terms for an arithmetic basket put option and $d = 2$. We choose $\boldsymbol{N} = (n, n)$ and $\boldsymbol{M} = \boldsymbol{L} = (\frac{1}{2}n^{\beta}, \frac{1}{2}n^{\beta})$ with $\beta = \frac{1}{2}$. Theor. slope $-(1-\beta)\left(\frac{2T}{\nu} - \frac{d}{2}\right)$.

Damping factor



Error of the price of a cash-or-nothing put option in the BS model for $\boldsymbol{M} = \boldsymbol{L} = (20\sigma, ..., 20\sigma)$ and $\boldsymbol{N} = (70, ..., 70)$.

MC COS



CPU time of the COS method (τ_{COS}) and the CPU time of a MC simulation (τ_{MC}) for the BS model pricing a cash-or-nothing put option.

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