iCOS: Option-Implied COS Method

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International Conference on Computational Finance
CWI, Amsterdam

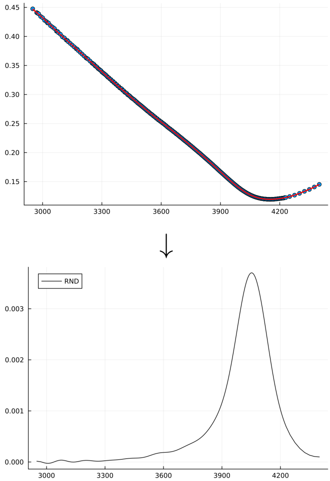
2 April 2024

Introduction

- Option price (panel) data contain valuable information about investor expectations on future asset prices;
- From option prices, one can obtain information on the (risk-neutral) density, moments, volatility, etc;
- This rich information, however, is often challenging to extract
  - *the non-linear nature of option prices and various sources of uncertainty add complexity and computational costs to the analysis*

RND from options

- Observed option prices convert to BSIV
- BSIV ‘curve-fitting’ (parametric or non-parametric)
- Convert the curve back to options
- Obtain the RND by calculating the second-order derivative of a (converted) ‘curve’.



In this paper

- We propose a unified non-parametric estimation procedure for the RND, option prices and option sensitivities (option delta);
- Our approach leverages the Fourier-based cosine technique, the COS method, proposed by Fang and Oosterlee (2008), in a model-free way by *implying* information from observed option contracts.
- The proposed estimation method is fully non-parametric and does not require any optimization routines, offering a flexible and computationally appealing alternative to traditional techniques.

The COS method

- The COS method of Fang and Oosterlee (2008) is the Fourier-based based method for evaluation of option prices;
- Fourier cosine series expansion of the density function $f(y)$ on an interval $[a, b] \subset \mathbb{R}$:

$$f(y) = \frac{2}{b-a} \sum_{m=0}^{\infty}{}' A_m \cos\left(u_m y - u_m a\right), \tag{1}$$

with $u_m := \frac{m\pi}{b-a}$ and the cosine coefficients A_m:

$$A_m = \int_a^b \cos(u_m y - u_m a) f(y) \mathrm{d}y. \tag{2}$$

The COS method

- *Fang and Oosterlee (2008):* the cosine coefficients A_m can be approximated via the characteristic function (CF), $\phi(u)$, of y:

$$A_m = \Re\left\{\int_a^b e^{\mathrm{i}u_m(y-a)} f(y)\mathrm{d}y\right\} \approx \Re\left\{\phi(u_m)e^{-\mathrm{i}u_m a}\right\} =: \widetilde{A}_m.$$

- Define the cosine series coefficients of the payoff function $v(y,T)$ as

$$H_m := \frac{2}{b-a}\int_a^b v(y,T)\cos(u_m y - u_m a)\mathrm{d}y.$$

  - *deterministic function of the payoff function, pre-computed for put and call.*

The COS method

- Consider a European option with a payoff $v(y,T)$ maturing at time T;
- Its price under the risk-neutral measure $\mathbb{Q}$ is given by the COS as:

$$v_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[v(y,T)] \overset{(1)}{\approx} e^{-rT}\int_a^b v(y,T)f(y)\mathrm{d}y$$
$$\overset{(2)}{\approx} e^{-rT}\sum_{m=0}^{\infty}{}' \widetilde{A}_m H_m \overset{(3)}{\approx} e^{-rT}\sum_{m=0}^{N-1}{}' \widetilde{A}_m H_m,$$

where by $\overset{(i)}{\approx}$ we denote the subsequent numerical approximation.

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where by $\overset{(i)}{\approx}$ we denote the subsequent numerical approximation.
- Importantly, it requires **a parametric** CF, which is unknown a priory;
- However, we observe option prices in the market!

Information in options

- *Breeden and Litzenberger (1978):* call (and put) option on underlying asset price S_T with strike K and time to expiration T:

$$C_0(K) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\max(S_T - K, 0)] = e^{-rT}\int_K^{\infty}(x-K)f_S(x)\mathrm{d}x,$$

implies risk-neutral density

$$f_S(x) = e^{rT}C_0''(x) = e^{rT}\left.\frac{\partial^2 C_0(K)}{\partial K^2}\right|_{K=x}. \tag{3}$$

Payoff spanning

- Consider a general payoff function $v(S_T)$ for a European-style option
- Value v_0 at time $t = 0$ for this option is

$$v_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[v(S_T)] = \int_0^\infty v(x) f_S(x) \mathrm{d}x.$$

- *Carr & Madan (2001):* replicating portfolio of this contract:

$$v_0 = e^{-rT} v(F_0) + \int_0^\infty v''(K) O_0(K) \mathrm{d}K$$

with OTM options $O_0(K) = \min\{C_0(K), P_0(K)\}$ and futures price F_0.

Option-implied cosine coefficients

Transformation $y = \log \frac{S_T}{x}$, $a = \log \frac{\alpha}{x}$ and $b = \log \frac{\beta}{x}$ with $x > 0$, and notice

$$A_m = \int_a^b \cos\left(u_m y - u_m a\right) f(y) \mathrm{d}y \tag{4}$$

$$= \int_\alpha^\beta \cos\left(u_m \log \frac{S_T}{\alpha}\right) f_S(S_T) \mathrm{d}S_T = \mathbb{E}^{\mathbb{Q}}\Big[\underbrace{\cos\left(u_m \log \frac{S_T}{\alpha}\right) \mathbf{1}_{\{\alpha \le S_T \le \beta\}}}_{=:v(S_T)}\Big],$$

which can be spanned as a portfolio of options with $v(S_T)$:

$$e^{-rT} A_m = \underbrace{e^{-rT} \cos\left(u_m \log \frac{F}{\alpha}\right) + \int_\alpha^\beta \psi_m(K) O_0(K) \mathrm{d}K}_{=:D_m} + \underbrace{(-1)^m C'_K(\beta) - P'_K(\alpha)}_{=:b_m}.$$

Option-implied cosine coefficients

- $C'_K(\beta)$ and $P'_K(\alpha)$ are call and put price derivatives w.r.t. strike K and $\psi_m(x) := v''(x)$
- The option-implied cosine coefficients D_m are exact and completely model-free;
- But, require a continuum of options with strikes $K \in [\alpha, \beta]$;
- D_m can be estimated by $\widehat{D}_m$ given n observed OTM option prices $O(K_i)$, $i = 1, \ldots, n$ with $\alpha := K_1 < ... < K_n =: \beta$.
- In fact, $\widehat{D}_m \xrightarrow{P} D_m$ as $n \to \infty$.

Option-implied call price

The call price $C_0(x)$ with the strike price x: $\alpha \leq x \leq \beta$ can be decomposed as

$$C_0(x) = C_0^{[\alpha,\beta]}(x) + C_0^{(\beta,\infty)}(x) = C_0^{[\alpha,\beta]}(x) + (x-\beta)C_K'(\beta) + C_0(\beta),$$

- $C_0^{[\alpha,\beta]}$ is the value of the call on the interval $[\alpha, \beta]$;
- $\Rightarrow$ can be evaluated exactly using the COS expansion:

$$C_0(x) = {\sum_{m=0}^{\infty}}' D_m H_m(x) + C_0(\beta) + Z_c(x)C_K'(\beta) + Z_p(x)P_K'(\alpha). \tag{5}$$

- $Z_c(x)$ and $Z_p(x)$ some deterministic functions of x;
- (5) is exact but circular $\Rightarrow$ useful for interpolation/approximation;

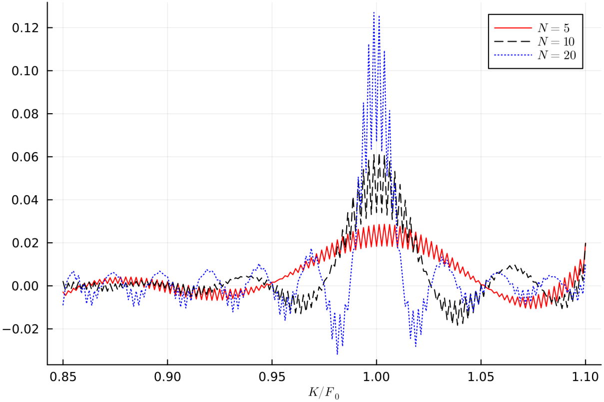
In practice

1. Given a finite number of option prices, estimate $\widehat{D}_m$, $m = 1, \ldots, N$;
2. estimate $\widehat{\overline{C}}(x) := \sum_{m=0}^{\prime N} \widehat{D}_m H_m(x)$ for $x \in [\alpha, \beta]$;
3. regress $C(K_i) - \widehat{\overline{C}}(K_i) - C(\beta)$ on $Z(K_i) := (1, Z_c(K_i), Z_p(K_i))$ to get $\widehat{\theta} := \left(\widehat{\overline{\theta}}, \widehat{C}_K'(\beta), \widehat{P}_K'(\beta)\right)'$;
4. obtain $\widehat{C}(x) := \widehat{\overline{C}}(x) + C(\beta) + Z(x)\widehat{\theta}$ for any strike $x \in [\alpha, \beta]$.

Option-implied call price

- The call price estimator $\widehat{C}(x)$ is model-free and eliminates approximation errors (1) and (2);
- In contrast to the COS method, the choice of the interval $[a, b]$ is data-driven;
- $\widehat{C}(x)$ is a portfolio of all observed put/call options
  $\Longrightarrow$ can be used for smoothed interpolation

Portfolio weights for ATM call option



The illustration is based on the simulated Black-Scholes model with strike prices between 85% and 110% of the spot price with equidistant increments and 201 option contracts.

Assumptions

1. The smallest and largest strike prices are fixed at $\alpha = K_1$ and $\beta = K_n$, and the strike prices between them are equidistant, i.e. $\Delta K := K_i - K_{i-1} = \frac{\beta-\alpha}{n-1}$, $i = 2, \ldots, n$.
2. Option prices are observed with an additive error term:

$$O(K_i) = O_0(K_i) + \varepsilon_i, \quad i = 1, \ldots, n,$$

   where ε_i are mean-zero, conditionally independent, and heteroskedastic across strikes.
3. The RND of future prices $f_S(s) \in C^p([\alpha, \beta])$ with $p > 1$.

Option-implied call

Proposition

Under Assumptions 1–3, the computationally feasible option-implied call price estimator $\widehat{C}(x)$ with a strike price $x \in [\alpha, \beta]$ is such that as $n \to \infty$ and $N \to \infty$ with $Nn^{-1/2} \to 0$

$$\frac{\widehat{C}(x) - C_0(x)}{\sigma_c(x)} \xrightarrow{d} \mathcal{N}(0, 1),$$

with a closed-form expression for the variance $\sigma_c^2(x)$.

Risk-Neutral Density

Proposition

Under Assumptions 1–3, the computationally feasible option-implied RND estimator of the future log price $y = \log S_T$*:*

$$\widehat{f}(y) = \nu_f {\sum_{m=0}^{N}}' \left(\widehat{D}_m + (-1)^m \widehat{C}'_K(\beta) - \widehat{P}'_K(\alpha) \right) \cos\left(u_m y - u_m \log \alpha \right) \tag{6}$$

with $\nu_f := 2e^{rT} \log\left(\alpha / \beta\right)$ *and is such that for any fixed* y *as* $n \to \infty$ *and* $N \to \infty$ *with* $Nn^{-1/6} \to 0$

$$\frac{\widehat{f}(y) - f(y)}{\nu_f \sigma_f(y)} \xrightarrow{d} \mathcal{N}(0,1),$$

where $\sigma_f^2(y)$ *is the variance term.*

Option-implied delta

The implied, model-free delta for a European call can also be obtained via a portfolio of options:

$$\Delta(x) = \frac{\partial C_0(x)}{\partial S_0} = -\frac{1}{S_0}\sum_{m=1}^{\infty} u_m B_m H_m(x) + \frac{1}{S_0}\left(C_0(\beta) - \beta C_K'(\beta)\right).$$

- Coefficients $B_m := e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\sin\left(u_m \log\frac{S_T}{\alpha}\right)\mathbf{1}_{\{\alpha \le S_T \le \beta\}}\right]$ are spanned in a similar way as A_m;
- The non-parametric estimator for the option's Delta can also be derived similarly:

$$\widehat{\Delta}(x) = -\frac{1}{S_0}\sum_{m=1}^{N} u_m \widehat{B}_m H_m(x) + \frac{1}{S_0}\left(C(\beta) - \beta\widehat{C}_K'(\beta)\right); \qquad (7)$$

- An analogous asymptotic result holds for the delta estimator $\widehat{\Delta}(x)$.

iCOS

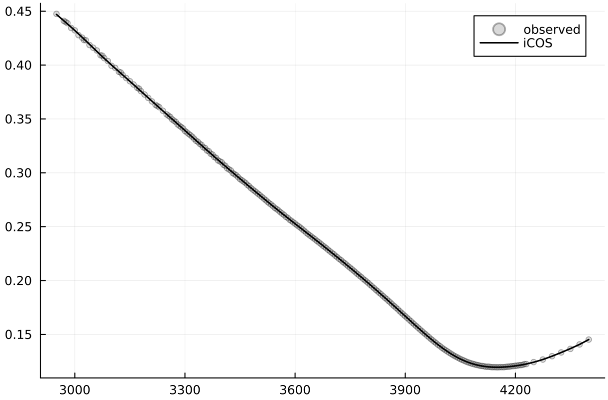
- iCOS combines the COS machinery with the *option-implied* information;
- non-parametric and based on portfolio replication argument
  - *does not do 'curve-fitting' and does not use Breeden and Litzenberger (1978);*
  - *(almost) optimization-free;*
- uses all available information
  - *kernel smoothing and local regression models use local information controlled by bandwidth parameter;*
- the simulation results indicate good finite performance more;
- choice of N can be adaptive more.

S&P 500 options

- SPX options traded on April 1, 2021 with maturity $T = 29$ days;
- Focus on option with strikes from interval $[\alpha, \beta] = [2950, 4400]$, while forward $F_0 = \$4008.5$;
- Use mid-quote option prices as the input for the estimation procedure;
- The number of options $n = 239$ and Fourier terms $N = 23$.

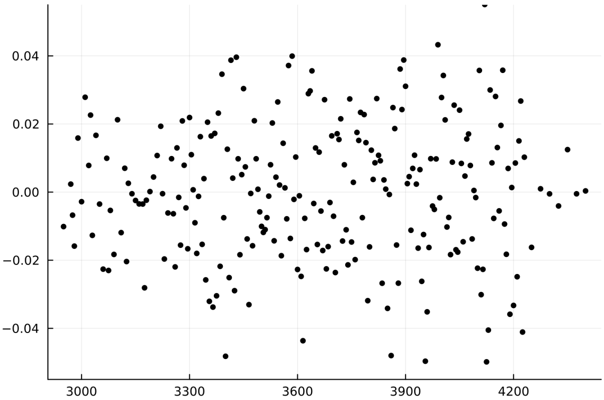
S&P 500 options

Figure 1: SPX options fit displayed on BSIV space



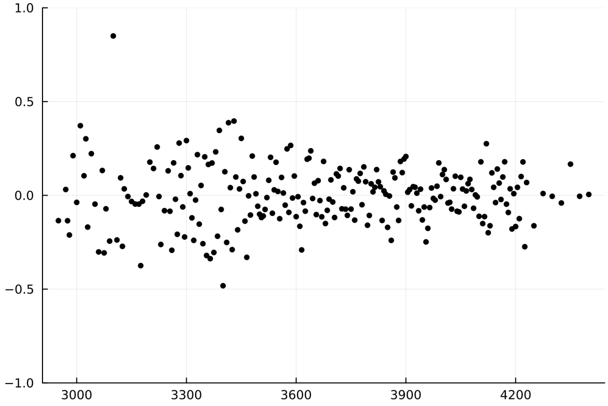
S&P 500 options

Figure 2: Pricing errors for call estimates $\widehat{C}(K) - C(K)$



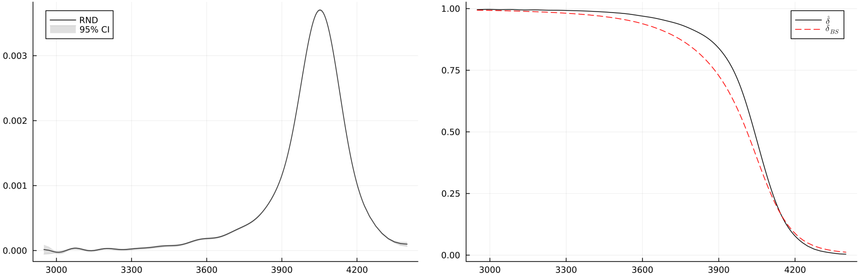
S&P 500 options

Figure 3: Pricing errors for call estimates $\widehat{C}(K) - C(K)$ relative to the half-spread



S&P 500 options

Figure 4: RND and call delta for SPX options

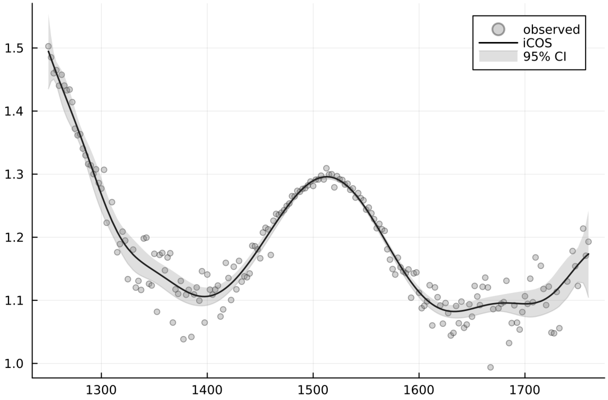


Amazon options

- Equity options on Amazon with a very short time-to-maturity of 1 day;
- Earning Announcement Day (EAD) of April 26, 2018 (before the announcement);
- Atypical W-shaped implied volatility curve and bimodal RND
    - *much harder to capture with standard curve-fitting methods*

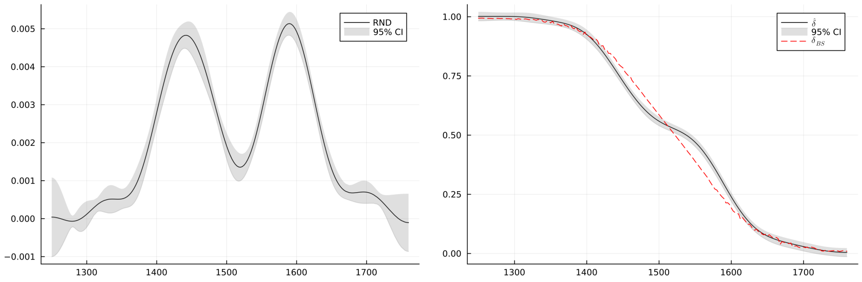
Amazon options

Figure 5: AMZN options fit displayed on BSIV space



Amazon options

Figure 6: RND and call delta for AMZN options



Conclusion

- Develop a new non-parametric estimation procedure for the RND, option prices and deltas;
- Based on the COS method with the option-implied information;
- It is model-free, non-parametric and does not require optimization procedures;
- Simulation results suggest good finite-sample performance;
- Application to SPX options and AMZN equity option on EAD.

Simulation setup back

- The 'double-jump' stochastic volatility model of Duffie, Pan, and Singleton (2000):

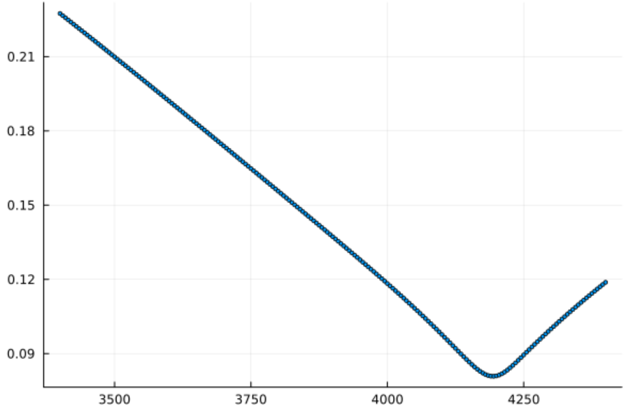
$$\begin{aligned} \mathrm{d}\log S_t &= (r - \tfrac{1}{2}v_t - \mu\lambda)\mathrm{d}t + \sqrt{v_t}\mathrm{d}W_{1,t} + J_t\mathrm{d}N_t, \\ \mathrm{d}v_t &= \kappa(\bar{v} - v_t)\mathrm{d}t + \sigma\sqrt{v_t}\mathrm{d}W_{2,t} + J_t^v\mathrm{d}N_t. \end{aligned}$$

- $T = 30$ days and the number of options $n = 201$;
- Add errors into the simulated option prices:

$$O(K_i) = O_0(K_i) + 0.025 \cdot \epsilon, \quad i = 1, \ldots, n,$$

where ϵ is an i.i.d. standard normal random variable.

Simulated option prices



Simulated option prices from the SVCJ model displayed on BSIV. The option prices are simulated using the COS method given the analytic solution for the CCF and a large number of expansion terms $N = 1024$ with $[a, b] = [-4\sqrt{T}, 4\sqrt{T}]$.

Simulation setup

- The number of expansion terms is $N = 25$.
- Compare with the kernel smoothing method with the following bandwidth:

$$h = \frac{c}{\log n} n^{-\frac{1}{2p+1}}$$

with $p = 2$ and different constant values $c > 0$.

Simulation results. Call prices

Table 1: Monte Carlo results for the call prices under the SVCJ model

	K/F_0	0.86	0.9	0.95	1.0	1.05	1.09
$C_0(K)$		560.66	402.23	210.81	54.13	0.61	0.14
iCOS	MC bias	-0.00075	0.00039	-0.00036	3.0e-5	-0.00207	-0.00055
	MC std	0.0102	0.00969	0.00941	0.00926	0.009	0.00933
	As. std	0.00981	0.00944	0.009	0.00909	0.00906	0.00893
KS, $c = 0.2$	MC bias	-0.07909	-0.00276	-0.00383	-0.10338	0.35918	-0.04309
	MC std	0.00442	0.00427	0.0044	0.00433	0.00637	0.00369
KS, $c = 0.1$	MC bias	-0.01325	2.0e-5	-0.00027	-0.02496	0.11357	-0.00912
	MC std	0.00621	0.00582	0.00603	0.00589	0.00661	0.00594
KS, $c = 0.05$	MC bias	-0.00078	0.0005	6.0e-5	-0.00627	0.03224	-0.00178
	MC std	0.00828	0.00811	0.00835	0.00831	0.00839	0.00834
KS, $c = 0.03$	MC bias	-0.00066	0.00068	0.00024	-0.00249	0.01229	-0.00133
	MC std	0.01043	0.01051	0.01063	0.01061	0.01068	0.01082

Simulation results. RND back

Table 2: Monte Carlo results for the RND under the SVCJ model

	K/F_0	0.86	0.9	0.95	1.0	1.05	1.09
$f(\log K)$		0.13	0.49	2.76	11.37	3.11	0.03
iCOS	MC bias	0.0211	0.0032	0.0058	-0.0066	0.0711	0.0216
	MC std	0.1503	0.1138	0.0994	0.0972	0.0954	0.0937
	As. std	0.1462	0.1099	0.095	0.0933	0.091	0.0854
KS, $c = 0.2$	MC bias	0.002	0.0511	0.0169	0.319	-0.2341	0.0161
	MC std	0.0033	0.005	0.006	0.0702	0.0075	0.0043
KS, $c = 0.1$	MC bias	-0.0999	-0.0373	-0.1284	-0.2761	-0.1874	-0.0584
	MC std	0.0211	0.0263	0.0286	0.0316	0.0327	0.0285
KS, $c = 0.05$	MC bias	-0.0209	-0.0203	-0.0609	-0.1186	-0.1087	-0.0186
	MC std	0.1364	0.1469	0.1541	0.1627	0.1743	0.1844
KS, $c = 0.03$	MC bias	0.0063	-0.0174	-0.0296	0.0006	-0.0703	-0.0091
	MC std	0.469	0.4929	0.5117	0.5418	0.5904	0.6125

Cosine coefficients

The cosine coefficients D_m are estimated as

$$\widehat{D}_m := e^{-rT}\cos\left(u_m \log\frac{F}{\alpha}\right) + \sum_{i=1}^{n} w_i \psi_m(K_i) O(K_i)\Delta_n, \tag{8}$$

where w_i are the coefficients of a chosen numerical integration method.

Proposition

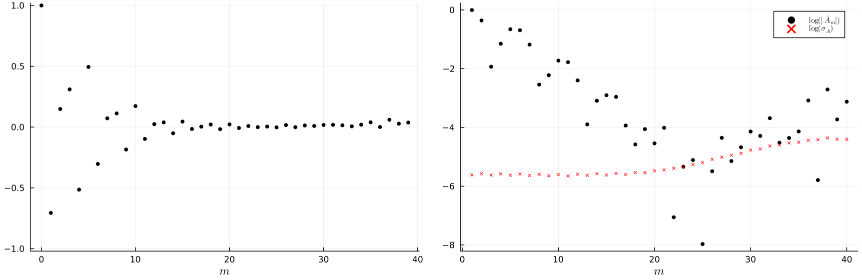
Under Assumptions 1–2, $\mathbb{E}\left[\widehat{D}_m - D_m\right] = \zeta^D_{m,n}$, *where* $\zeta^D_{m,n}$ *is the discretization error with the order depending on the chosen numerical integration scheme, and as* $n \to \infty$

$$\frac{\widehat{D}_m - D_m}{\sigma_D(m)} \xrightarrow{d} \mathcal{N}(0,1),$$

with $\sigma^2_D(m) = \sum_{i=1}^n w_i^2 \psi_m^2(K_i)\sigma_i^2 \Delta_n^2$.

S&P 500 options back

Figure 7: Estimated cosine coefficients $\widehat{D}_m$ (left) and $\widehat{A}_m$ (right)



References I

Breeden, D. T., & Litzenberger, R. H. (1978). Prices of state-contingent claims implicit in option prices. *Journal of Business*, 621–651.

Duffie, D., Pan, J., & Singleton, K. (2000). Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, *68*(6), 1343–1376.

Fang, F., & Oosterlee, C. W. (2008). A novel pricing method for European options based on Fourier-cosine series expansions. *SIAM Journal on Scientific Computing*, *31*(2), 826–848.