# Quantitative modelling and analysis of the Automated Market Maker Uniswap

Mnacho Echenim Emmanuel Gobet Anne-Claire Maurice

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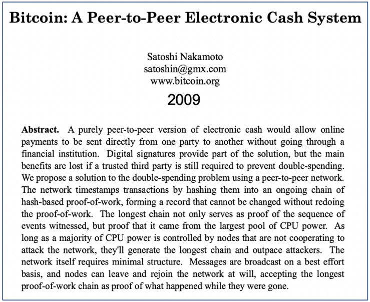
# The context of crypto-assets

Tutorial: SIAM FME Philadelphia, June 2023, [GM23]

## ▷ Bitcoin's Goal



**Bitcoin: A Peer-to-Peer Electronic Cash System**

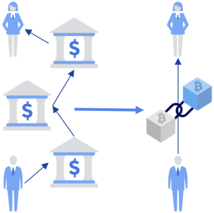
Satoshi Nakamoto
satoshin@gmx.com
www.bitcoin.org

2009

**Abstract.** A purely peer-to-peer version of electronic cash would allow online payments to be sent directly from one party to another without going through a financial institution. Digital signatures provide part of the solution, but the main benefits are lost if a trusted third party is still required to prevent double-spending. We propose a solution to the double-spending problem using a peer-to-peer network. The network timestamps transactions by hashing them into an ongoing chain of hash-based proof-of-work, forming a record that cannot be changed without redoing the proof-of-work. The longest chain not only serves as proof of the sequence of events witnessed, but proof that it came from the largest pool of CPU power. As long as a majority of CPU power is controlled by nodes that are not cooperating to attack the network, they'll generate the longest chain and outpace attackers. The network itself requires minimal structure. Messages are broadcast on a best effort basis, and nodes can leave and rejoin the network at will, accepting the longest proof-of-work chain as proof of what happened while they were gone.

Source: Nakamoto, S. (2008). Bitcoin whitepaper. URL: https://bitcoin. org/bitcoin. pdf-(: 17.07. 2019).

**What is bitcoin about ?**
Bitcoin is a peer-to-peer form of digital cash, enabling one individual to **transfer value** to another digitally **without the need for specific third-party intermediaries** to approve or deny the transaction.



A Revolution for trust, privacy, and security.

**Bitcoin = a technology and a digital currency**

# About Blockchain and its Evolution

## Bitcoin

Bitcoin allows transfers between users. It is like having a software that can do only one thing: **transactions**

Would be possible to have more features ? To allow more actions than simple transactions ?

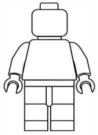
## Ethereum

The Ethereum blockchain was born a few years after bitcoin, in **2014**.

The native crypto of that chain is called ether **(ETH)**.

Ethereum also allows direct **transactions** between users, but there is a major change: **Ethereum allows computer programs to run directly on the blockchain !**

What is a smart contract?



**Hey, I'm a smart contract.**

**I promise you the following**

1. **Immutable.** I will never modify or change your code.
2. **Authenticated calls.** I will always run the function you tell me too (assuming the code allows me!).
3. **Atomic.** I will never let code execution "stop half way" it is ALL or NOTHING with me.
4. **No privacy.** I like to gossip and I can't keep secrets - Everything you tell me will be public knowledge.

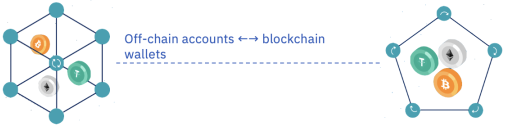
Other examples of Blockchain: Binance Smart Chain, Polygon, Avalanche, Tezos . . .

# The Blockchain, now at https://txstreet.com

# Differences between CEX and DEX

✓ CEX=Centralized Exchanges (off-chain): Coinbase, Bitstamp, Binance, Gemini, Kraken ··· - See Kaiko Exchange Ranking

✓ DEX=Decentralized Exchanges (on-chain): smart contracts ···




**Centralized Exchanges (CEXs)**

- Centrally managed (off-chain)
- Exchange controls your assets
- Listing fees and due diligence for new trading pairs
- Regulatory jurisdictions and KYC
- Barriers to market making
- Uses an order book
- You cannot trade if the exchange is down

**Decentralized Exchanges (DEXs)**

- Decentralized protocol (on-chain)
- You control your assets
- Anyone can list a pair by creating a new liquidity pool
- No regulation or KYC
- Anyone can be a market maker
- Automated Market Maker
- Data recorded directly on the blockchain

**About Price Discovery, how to replace Order-Book in CEX by ??? in DEX without specific third-party intermediaries?**

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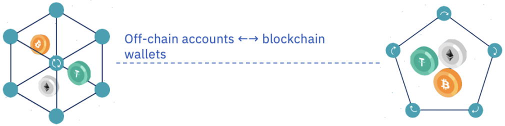


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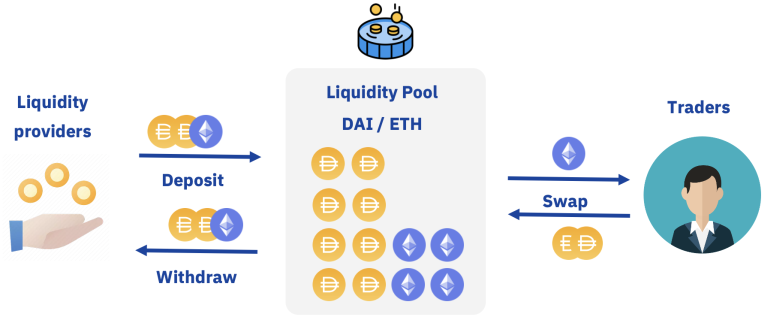
# Automated Market Maker (AMM)

## Pools based DEXs General Functioning




- ✓ The amount of tokens for each trade is decided by the **algorithm of the pool**
- ✓ Rules according to a **Constant Function Market Making**
- ✓ Exemples: Balancer, Curve, mStable, Sushiswap, Uniswap, etc
- ✓ Transactions are on-chain

# A few statistics on DEX - AMM

| # Trades | Exchange | # Pools | # mints | # Burns | % # Pool |
|---|---|---|---|---|---|
| 114814668 | usp2 | 195403 | 3924111 | 1740615 | 90,03% |
| 32452350 | usp3 | 13037 | 1108823 | 1242071 | 6,01% |
| 11976425 | sush | 3684 | 531695 | 308206 | 1,70% |
| 2787827 | blcr | 3291 | 305293 | 200417 | 1,52% |
| 1398792 | blc2 | 1275 | 93864 | 44279 | 0,59% |
| 960924 | curv | 344 | 566152 | 421482 | 0,16% |
| 471382 | crv2 | 18 | 18261 | 11276 | 0,01% |

Figure: Number of trades and liquidity pools, by DEX, on Ethereum

## TALK AGENDA

- ✓ Quick overview on Uniswap v2
- ✓ Our contributions on Uniswap v3:
  - ✓ Formalize LPs curves and payoffs for understanding automated market-making mechanisms
  - ✓ Study impermanent loss for LPs, links with options
  - ✓ Refine LPs returns by considering transaction fees in the analysis, asymptotic formula
- ✓ Perspectives and open questions

# Diving into Uniswap v2 - Principles

✓ Based on the Constant Function Market Makers:

$$\mathcal{I}(x, y) = x \cdot y.$$

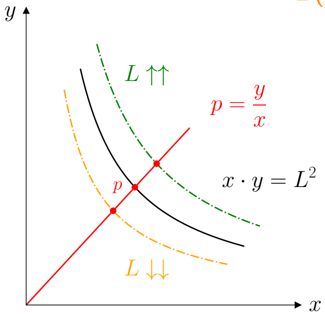



Figure: How prices and quantities evolve in a Uniswap v2 protocol

✓ Price/tokens/liquidity relations:

$$x = \frac{L}{\sqrt{p}} = \frac{L}{\pi} \quad \text{and} \quad y = L\sqrt{p} = L \cdot \pi.$$

where $p$ is the pool price and $\pi$ is the square root price

# Impermanent loss

- ✓ Liquidity providers: rewarded for depositing their tokens in a pool by the swap fees
- ✓ If LPs withdraws their tokens at $t$ where $p_t \neq p_0$, it may result in a lower global value (net of fees)
  $\Rightarrow$ *Impermanent Loss*, or *Divergence Loss*. See [Pin20].

Compare two strategies:

- ✓ HODL strategy: $V_H$ = the value of a portfolio where tokens are held in a separate wallet
- ✓ Liquidity providing strategy: $V_P$ = the value of a portfolio where tokens are invested in a liquidity Uniswap v2 pool.

As a convention we take $Y$ as *reference numéraire*.

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$$\mathrm{IL} = V_P - V_H.$$

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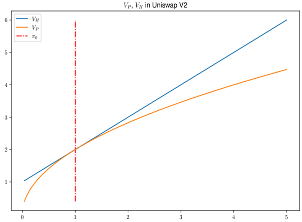
# Uniswap v2 - IL Computation (neglecting fees)

From

$$\begin{aligned} V_H &= \Delta x_0 \cdot p_1 + \Delta y_0 \\ &= \Delta x_0 \cdot p_1 + \Delta x_0 \cdot p_0, \\ V_P &= x_1 \cdot p_1 + y_1 \\ &= \frac{\Delta x_0 \cdot \pi_0}{\pi_1} \cdot p_1 + (\Delta x_0 \cdot \pi_0) \cdot \pi_1 \\ &= 2 \cdot \Delta x_0 \cdot \pi_0 \cdot \pi_1, \end{aligned}$$




we deduce

$$V_P - V_H = 2 \cdot \Delta x_0 \cdot \left( \sqrt{p_0 \cdot p_1} - \frac{1}{2}(p_0 + p_1) \right) \leq 0.$$

- ✓ Without swap fees, no P&L interest for the LP
- ✓ The pool behaves as an option (non linear contract) with arbitrary maturity and concave payoff (like shorting calls/puts)
- ✓ LP position is Gamma negative and $\theta$ positive (collected fees)

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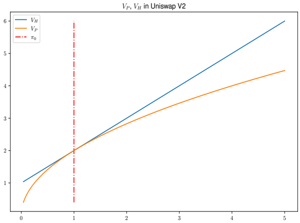


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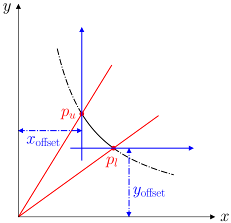
# Diving into Uniswap v3 - Principles

- ✓ **Concentration of liquidity.**
  - ✓ LP provides liquidity on a chosen price range $R := [p_\ell, p_u)$
  - ✓ In practice: some tokens $X$ and $Y$ in that range
- ✓ **Swap trades.**
  - ✓ Tokens in a range $[p_\ell, p_u)$ are useable only when $p \in [p_\ell, p_u)$
  - ✓ When $p \downarrow p_\ell$ (resp. $\uparrow p_u$), reserves of tokens $Y$ (resp. $X$) are depleted.
- ✓ **Localized Constant Function Market Maker.**

On the price range $R := [p_\ell, p_u)$, we have $x \cdot y = L_R^2$ but $(x, y)$ represent *virtual* numbers of tokens $\begin{cases} x = x_r + x_{\text{offset}}, \\ y = y_r + y_{\text{offset}}, \end{cases}$




- ✓ $(x_r, y_r)$ are *real* numbers of tokens
- ✓ $(x_{\text{offset}}, y_{\text{offset}})$ are *offset* numbers (for boundary conditions)

$$\Longrightarrow \left(x_r + \frac{L_R}{\pi_u}\right) \cdot (y_r + L_R \cdot \pi_\ell) = L_R^2.$$

# Example of Liquidity curve

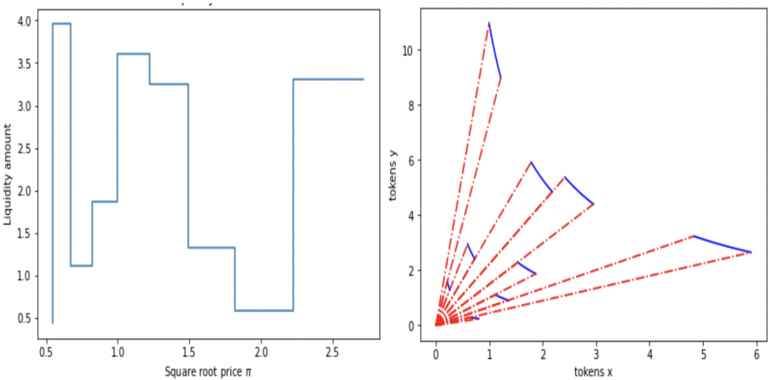



Figure: On the left: the liquidity curve.
On the right: the CPMM relation on each range.

# Details about liquidity range and fees

- ✓ For liquidity events, the ranges of prices are restricted to *ticks*:

$$p(\tau) := (\beta_{\mathrm{p}})^{\tau} \quad \text{where} \quad \beta_{\mathrm{p}} = 1.0001.$$

- ✓ An LP specify a tick range on which liquidity is to be added.
- ✓ Not all tick ranges can actually be used to update liquidity: the ranges are a multiple of a fixed number of ticks $\delta_\pi$, which is ruled at the setting of the pool, according to the swap fees $\phi$.

$\delta_\pi = 2, 10, 60, 200,$ according to swap fees $\phi = 0.01\%, 0.05\%, 0.3\%, 1\%$

| Transaction Fee | Uniswap V2 # Pools | Uniswap V3 # Pools | Total # pools | Share Uniswap V3 |
|---|---|---|---|---|
| 0.0001 | | 96 | 96 | 1% |
| 0.0005 | | 300 | 300 | 4% |
| 0.003 | 4842 | 1149 | 5991 | 79% |
| 0.01 | | 1212 | 1212 | 16% |
| **Total** | **4842** | **2757** | **7599** | |

- ✓ When a range is defined by two consecutive ticks $i \cdot \delta_\pi$ and $(i+1) \cdot \delta_\pi$, we refer to a *unitary range*.

# Our contributions on Uniswap v3

- ✓ Contributions:
  - ✓ Explicit value of the LP-position
  - ✓ Replication of concave payoff with Uniswap v3 pool
  - ✓ Asymptotics formula for computing/predicting fees
  - ✓ Full generality on liquidity events
- ✓ For the fees analysis, assumptions on the $X - Y$ exchange rate:

  $\mathbf{H}_{Ito}$: The *latent* pool price process $(p_t)_t$ follows an Itô dynamics

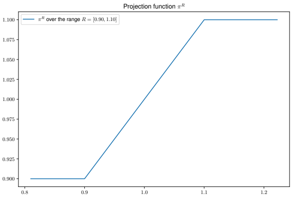
$$\frac{\mathrm{d}p_t}{p_t} = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t$$

  with possibly stochastic drift and volatility. All values are expressed using $Y$ as numeraire and pool price.
- ✓ Related litterature:
  - ✓ Optimal execution and deep neural networks: [JSST23]
  - ✓ Optimal liquidity provision: [CDM23], [FMCM+23], [CDSB+23], [FMCA+22]
  - ✓ Optimal trade: [CDM22]
  - ✓ Empirical study of Uniswap v3 pools: [LHRW21]
  - ✓ Impermanent Loss in Uniswap v3: [Lam21], [Bou22]
  - ✓ Fees: [BF23]
- ✓ Our analysis: from the open source code https://github.com/Uniswap/v3-core/

# Mathematical formulation of Mint/Burn by LP

Consider an *unitary square root price range* $[\pi_\ell, \pi_u)$ and define the projected square root price function:

$$\pi_0^R = \begin{cases} \pi_u \text{ if } \pi_u < \pi_0, \\ \pi_\ell \text{ if } \pi_0 < \pi_\ell, \\ \pi_0 \text{ otherwise.} \end{cases}$$




## Lemma (Relation between $\Delta L$ and $(\Delta x_r, \Delta y_r)$)

*The mint/burn operations (keeping the current square root price $\pi_0$ unchanged) on that range $R = [\pi_\ell, \pi_u)$ are described by a single formula that cover all situations*

$$\Delta x_r = \Delta L \cdot \left( \frac{1}{\pi_0^R} - \frac{1}{\pi_u} \right) \quad \textit{and} \quad \Delta y_r = \Delta L \cdot (\pi_0^R - \pi_\ell).$$

# Value of the LP position in the pool

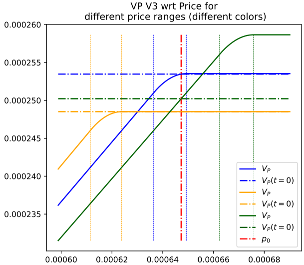
## Theorem ($V_P$ on a unitary range)

*Adding a liquidity $\Delta L$ on the unitary range $R = [\pi_\ell, \pi_u)$ at time 0 gives Y-value of the LP position equal to*

$$V_P(t) = \Delta L \cdot \left( \left( \frac{1}{\pi_t^R} - \frac{1}{\pi_u} \right) \cdot \pi_t^2 + (\pi_t^R - \pi_\ell) \right),$$

*given as a function of the square root price $\pi_t$ at time $t$.*

Referred as *Covered call* on Guillaume Lambert's blog [Lam21].




Value of the pool for 3 different price ranges (orange, blue, green). The dashed vertical lines in each color represent the considered unitary ranges in the variable price $p$. The dashed horizontal lines represent the initial value of the pool for the price $p_0$ depicted in red.

# Arbitrary liquidity curve

- ✓ Objective: better understand the link of the liquidity curve and the exposure to market changes

## Theorem ($V_P$ with an arbitrary liquidity curve)

*Consider a liquidity provider adding a liquidity curve $(\Delta L_\pi)_\pi$ to the pool at some times prior to $t$.*
*Then, its $Y$-value net of swap fees at time $t$ is*

$$V_P(t) = p_t \cdot \int_{\pi_t}^{+\infty} \frac{\Delta L_\pi}{\pi^2} d\pi + \int_0^{\pi_t} \Delta L_\pi d\pi$$

*when the pool price is $p_t$.*

- ✓ Remarkably simple!
- ✓ Valid for any occurence of swap trades or other mint/burn events.
- ✓ One can compute greeks. Allows for risk management of crypto portfolios.
- ✓ Gamma negative: $\Gamma_P(t=0) \stackrel{\text{def}}{=} \frac{\partial^2 V_P(t)}{\partial p_t^2} = -\frac{\Delta L_{\pi_t}}{2\pi_t^3}$.

# Implicit liquidity curve

## Theorem (Liquidity curve generating a given payoff)

*Consider a concave payoff $h : \mathbb{R}^+ \mapsto \mathbb{R}$ which we assume to be $C^3$ and linear for small and large values. Then, consider a strategy depositing the liquidity curve*

$$\Delta L_R := (-h''(\pi_\ell \cdot \pi_u)) \cdot (\pi_u + \pi_\ell) \cdot \pi_\ell \cdot \pi_u$$

*at time 0 on each range $R = [\pi_\ell, \pi_u)$. In addition, add to the position quantities $x_0$ of tokens $X$ and $y_0$ of tokens $Y$ outside the pool, with*

$$x_0 = h'(p_0) - \sum_{R=[\pi_\ell,\pi_u)} \Delta L_R \cdot \left(\frac{1}{\pi_0^R} - \frac{1}{\pi_u}\right),$$

$$y_0 = h(p_0) - h'(p_0) \cdot p_0 + \sum_{R=[\pi_\ell,\pi_u)} \Delta L_R \cdot \left(\pi_0^R - \pi_\ell\right).$$

*Then*

$$|h(p_T) - (V_P(T) + x_0 \cdot p_T + y_0)| \leq C \cdot \delta_\pi \cdot (\beta_{\mathrm{p}} - 1), \quad a.s.$$

*Recall that $\beta_{\mathrm{p}} = 1.0001$ (tick base).*

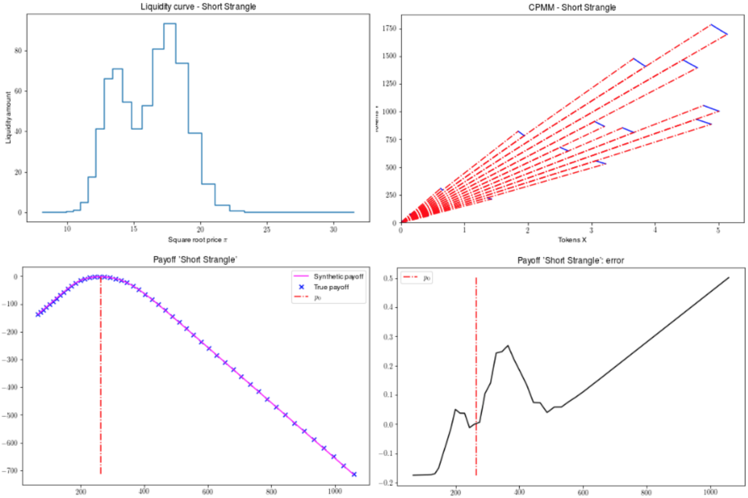



Figure: Illustration when $h$ corresponds to a short strangle (minus a put with strike $K = p_0/1.3$ and minus a call with strike $K = 1.3 \cdot p_0$. Both options are considered with a maturity $\tau = 0.1$ and a Black-Scholes volatility equal to 50%). Top left: the liquidity curve $\Delta L_R$. Top right: the CPMM representation. Bottom left: the payoff and its replication. Bottom right: the reconstruction error.

# Swap traders and fees bookkeeping

- ✓ Contrarily to Uniswap v2 pools, fees are not considered as additional tokens in the reserves of the Uniswap v3 pool.
- ✓ Fees are concentrated specifically on each range
- ✓ Given a range $R = [\pi_\ell, \pi_u)$, the *fees* in the pool are tracked by two accumulators $\Phi_R^X$ and $\Phi_R^Y$ that are updated at every transaction (`feeGrowthGlobal0X128` and `feeGrowthGlobal1X128` in the source code).
- ✓ Give amounts of fees per unit of liquidity.
- ✓ These accumulators are recovered by

$$\Phi_R^X = \Phi^X - \varphi_{\mathrm{b}}^X(\pi_\ell) - \varphi_{\mathrm{a}}^X(\pi_u) \quad \text{and} \quad \Phi_R^Y = \Phi^Y - \varphi_{\mathrm{b}}^Y(\pi_\ell) - \varphi_{\mathrm{a}}^Y(\pi_u),$$

which are implemented in the `getFeeGrowthInside` method of TICK.SOL and invoked when the position is updated (such as in the `_updatePosition` method of UNISWAPV3POOL.SOL).

# Analytical formula for fees

## Theorem (Fees in $X$ and $Y$ accumulated over $[0, T]$)

*Consider a LP depositing a liquidity curve $(\Delta L_\pi)_\pi$ at time 0. Assume*

- ✓ $(\Delta L_\pi)_\pi$ *has a finite support*
- ✓ $\mathbf{H}_{Ito}$ *and some mild assumptions on* $\mu_t, \sigma_t$
- ✓ *the swap trades occur as the latent price process $(p_t)_t$ moves from one tick to another*

*Then*

$$\lim_{\beta_{\mathrm{p}} \downarrow 1} (\beta_{\mathrm{p}} - 1) \cdot \mathrm{Fees}^X_{0 \to T} \stackrel{\mathbb{P}}{=} \frac{\phi}{1-\phi} \cdot \int_0^{+\infty} \Delta L_{b^{\frac{1}{2}}} \frac{A^b_T(p)}{4 \cdot b^{5/2}} \mathrm{d}b,$$

$$\lim_{\beta_{\mathrm{p}} \downarrow 1} (\beta_{\mathrm{p}} - 1) \cdot \mathrm{Fees}^Y_{0 \to T} \stackrel{\mathbb{P}}{=} \frac{\phi}{1-\phi} \cdot \int_0^{+\infty} \Delta L_{b^{\frac{1}{2}}} \frac{A^b_T(p)}{4 \cdot b^{3/2}} \mathrm{d}b,$$

*where $A^b_T(p)$ is the local time of $p$ at level $b$ and time $T$:*

$$\mathrm{d}(p_t - b)_+ = 1_{p_t \le b} \mathrm{d}p_t + \frac{1}{2} \mathrm{d}A^b_t(p).$$

# Confirmation from experiments on synth. data

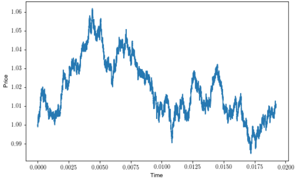



Figure: Sample path of GBM for $p$, with $p_0 = 1$, $\sigma = 40\%$ and drift $\mu = 5\%$

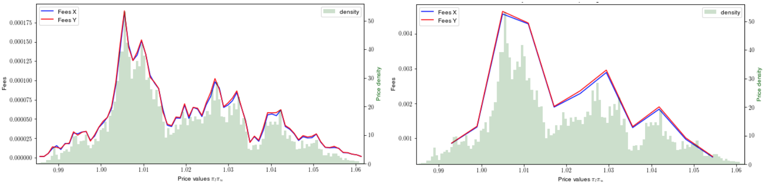



Figure: Left axis: the exact fees collected in tokens $X$ and $Y$. Right axis: occupation density of $p$. Left: when $\delta_\pi = 10$. Right: when $\delta_\pi = 60$.

Indeed, fees depend much on the local time!

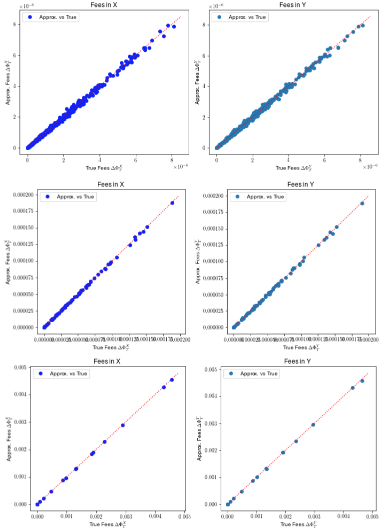



Figure: True collected fees versus their approximations. From top to bottom: $\delta_\pi = 2, 10, 60$. Each point corresponds to the amount of swap fee on a range; smaller values of $\delta_\pi$ have more ranges hence more points.

Approximate formulas match exact fees!

# Experiments on real data

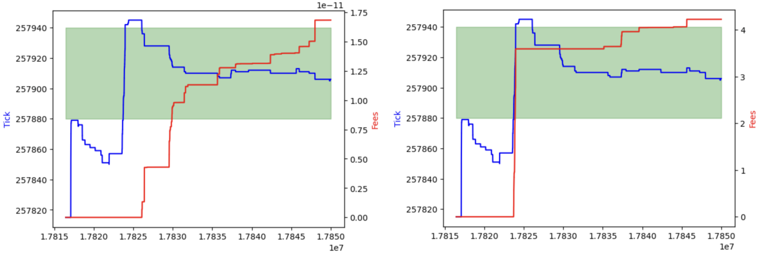
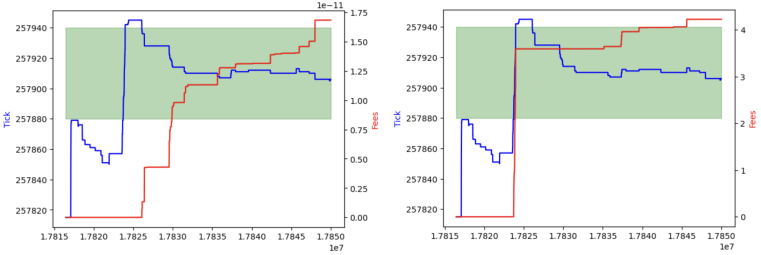



Figure: Uniswap v3 Pool: WBTC/WETH 0.3%, on Ethereum Blockchain. Time period: 33567 blocks (starting on 01/08/2023) ≈ 5 days. In blue: evolution of pool price (in ticks). On red: fees collected in tokens X and Y. Focus on a price range (of 60 ticks).

- ✓ On some periods, pool price behaves like an Ito process, and then fees evolve coherently with what the theory predicts
- ✓ In the other periods, the pool price does not stick to Ito dynamics (big trades, large moves/jumps. . . )
- ✓ Hopeless?

# Experiments on real data

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✓ Actually, results depend much on the blockchain used

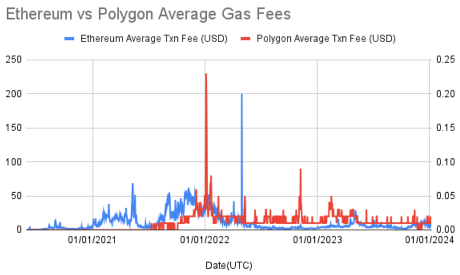


Figure: Comparison of Gas Fees on Ethereum and Polygon blockchains

✓ Smaller gas fees imply that orders can be more easily split ⇒ more frequent trades and liquidity events

| Blockchain | # Pools | # Trades | # Mints | # Burns | Since |
|---|---|---|---|---|---|
| Ethereum | 16711 | 43188321 | 1813109 | 2019288 | 01/05/2021 |
| Polygon | 10729 | 56913467 | 3637861 | 3821561 | 01/01/2022 |

Figure: Number of trades/burns/mints since origin (as of 9/1/2024)

✓ Extra tests in progress...

# Back to theoretical results...

## Theorem

*Consider the $Y$-value of the approximated collected swap fees:*

$$\mathfrak{F}((\Delta L_\pi)_\pi, (\sigma_t)_t) \stackrel{\text{def}}{=} \frac{1}{(\beta_{\mathrm{p}}-1)} \mathbb{E}^\star \Big[ \lim_{\beta_{\mathrm{p}}\downarrow 1} (\beta_{\mathrm{p}}-1)\cdot \mathrm{Fees}^X_{0\to T}\cdot p_T + \lim_{\beta_{\mathrm{p}}\downarrow 1} (\beta_{\mathrm{p}}-1)\cdot \mathrm{Fees}^Y_{0\to T} \Big].$$

*Assume the existence of a risk-neutral valuation rule under $\mathbb{P}^\star$ with unit discounted factor. We have*

$$(\beta_{\mathrm{p}}-1)\cdot \mathfrak{F}((\Delta L_\pi)_\pi, (\sigma_t)_t) = \frac{\phi}{(1-\phi)} \cdot \left( \int_0^{p_0} \Delta L_{b^{\frac{1}{2}}} \cdot \frac{\mathrm{Put}_{t=0}(T,b)}{b^{3/2}} \mathrm{d}b + \int_{p_0}^{+\infty} \Delta L_{b^{\frac{1}{2}}} \cdot \frac{\mathrm{Call}_{t=0}(T,b)}{b^{3/2}} \mathrm{d}b \right).$$

- ✓ Similar to a Carr-Madan formula
- ✓ Expected swap fees increase as volatility and maturity increase
- ✓ Allows for comparing fees with CEX options and studying CEX-DEX arbitrage

# Conclusion

## Our contributions on Uniswap v3

- ✓ Explicit value of the LP-position
- ✓ Replication of concave payoff with Uniswap v3 pool
- ✓ Asymptotics formula for computing/predicting fees
- ✓ Full generality on liquidity events
- ✓ *"Ito process assumption"* is questionable on Ethereum blockchain (large Gas Fees)

## Perspectives, open questions

- ✓ Generalizations of Proxy formula on other assumptions
- ✓ Statistical study of Uniswap v3 swap fees and comparison with the proxy formula for various blockchains
- ✓ Quantitative study of statistical arbitrage opportunities between CEX-DEX, options, spots
- ✓ Similar analysis for other AMMs

@PhD students, contact me if interested by PostDoc positions.

# Conclusion

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