Takers identification for defaulted portfolios with simulated annealing algorithms

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Outline

1. Motivation
2. Problem formulation
3. Theoretical aspects of Combinatorial Simulated Annealing
4. Numerical example
5. Conclusion

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Motivation (1 / 4)

This research looks at the problem of **identifying** multiple **takers** of **portfolios** of a financial actor **defaulting** on several **Central Clearinghouse Counterparties** [CCPs] based on the following principles:

- The **financial network** is *closed* (*adiabatic*): no input or output of cashflows other than those in the financial system
- If a default occurs, financial positions must be **re-allocated** in the system to ensure closedness

Scenario: a default happens on several CCPs that need to re-arrange positions with surviving members. Critical for financial **regulators** as large liquidity pools **concentrated by CCPs** and liquidity pressure default puts on other members.

CCP institutions:

- prominent financial actors from **G20 September 2009 summit** onward (G20 Research Group, 2009) with mandatory clearing for all "standardizable" (incl. OTC) products
- transform **counterparty risk** into **liquidity risk** by guaranteeing the negotiated contracts in exchange of various layers of **collaterals** posted by their members

Use Case 1: Ronin Capital (asset manager):

- default on two CCPs occurred in the United States
- in March 2020 during the Covid-19 pandemic crisis (illustrated in Bastide et al. (2023))

Use Case 2: Credit Suisse:

- beginning of 2023 liquidity issues whilst being a major member towards 30 CCPs across the world
- SNB granted a liquidity contribution of around €170bn (CHF 168bn, see Jordan (2023)) to prevent Credit Suisse from defaulting.

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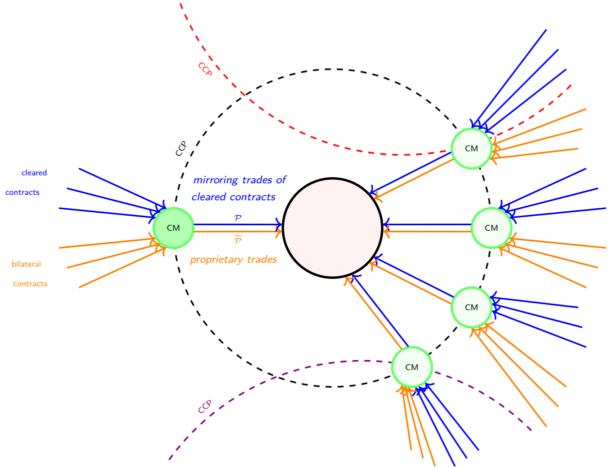
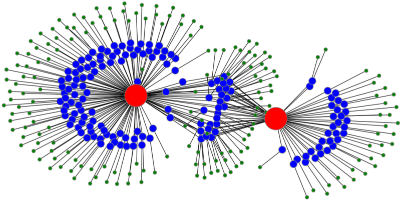
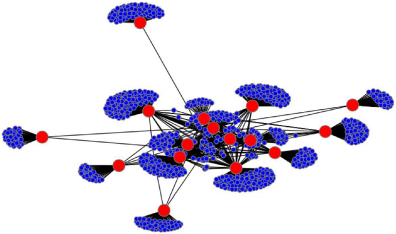


Figure 1: Promised cash flows between market participants. The reference clearing member bank is on the left.



(a) Network consisting of two CCPs (in red), 123 members for CCP1 seen on the left hand side, and 56 members for CCP2 on the right hand side, with 24 common members displayed as the group of members in the middle of the two CCPs (155 members in total, in blue), and with 179 cleared clients (in green).



(b) Large clearing network example (Q2-2021) with 16 CCPs in red and their members in blue with many having common memberships concentrated in the center of the network. Credit Suisse (EOY 2023) was one of the center blue dots.

Motivation (4 / 4) Link to simulated Annealing approach

CCP Problem characteristics:

- A member must be identified for each CCP the default has happen (recall Credit Suisse could have defaulted on 30 (major) CCPs)
- There are hundreds of major financial actors, many exposed to several CCPs at the same time
- The number of possible take-over configuration is of order 300^{50} (on reduced financial network)

Problem resolution approach: Given the high dimension of the problem and high computational costs,

- we develop the formulation to apply discrete simulated annealing algorithm technique to an approximated version of the problem
- we show theoretically that the solution should converge to the true solution of the true (non-approximated) problem
- we illustrate numerically that it can identify the set of optimal takers w.r.t. some cost minimization across all CCPs (represented by their members)
- we numerically show it outperforms a simple greedy search without re-sampling

Discrete simulated annealing applied in finance:

- Portfolio optimization through mean-variance formulation (Markowitz, 1952; Fabozzi et al., 2012; Rubio-García et al., 2022): Ingber (1993); Crama and Schyns (2003), combined with recent development of quantum computer technologies (Choi, 2011; Luoa et al., 2014; Lang et al., 2022)
- Business risk Eraña-Díaz et al. (2020)
- Reverse stress test (parameters selection) Montesi et al. (2020)

Discrete simulated annealing literature: Kirkpatrick et al. (1982, 1983); Aarts and van Laarhoven (1988); Aarts and Korst (1989); Catoni (1992); Duflo (1996); Henderson et al. (2003); Moral (2004); Delmas and Jourdain (2006); Delahaye et al. (2019)

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Problem formulation (1 / 2)

We assume an idealized auction with take-over members leading to the least costs to the entire network.

- K CCPs with member 0 defaulted and 10 (surviving) members exposed to all CCPs (if not, corresponding exposure is simply zero).
- $\mathcal{L}^{[k,i_k]}$ overall potential loss across members of CCP k post *default resolution* when i_k takes over the defaulted portfolio on CCP k.
- Loss allocated to members pro-rata to their position size with $\omega_\ell^{[k,i_k]}$ coefficient of member ℓ on CCP k post default resolution $\Rightarrow \mathcal{L}_\ell^{[k,i_k]} = \omega_\ell^{[k,i_k]} \mathcal{L}^{[k,i_k]}$
- Aggregated loss for member ℓ over all CCPs is $\sum_k \mathcal{L}_\ell^{[k,i_k]}$ belonging to some sub-vector space $\mathfrak{X} \subset \mathcal{L}^1(\Omega)$ defined from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- Each member minimizes risk metric $\mathcal{R}_\ell$ (e.g. expectation, quantile or expected shortfall) applied to its loss $\sum_k \mathcal{L}_\ell^{[k,i_k]}$ compensated by costs $c_\ell^{[\mathbf{i}]} \in \mathbb{R}$ (with $\mathbf{i} = (i_1, \ldots, i_K)$ and i_k taking over on CCP k).
- Takers $i_1^*, \ldots, i_K^* \in 1 \mathinner{..} L$ minimize the aggregated measure of risks over the CCPs' members

$$(i_1^*, \ldots, i_K^*) = \arg \min_{i_1, \ldots, i_K} \sum_{\ell=1}^{L} \mathcal{R}_\ell \left(\sum_{k=1}^{K} \mathcal{L}_\ell^{[k,i_k]} - c_\ell^{[i]} \right).$$

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- $\mathcal{L}_{\ell,c}^{[k,i_k]}$ is a function f_ℓ^k of some updated r.v.'s $\left(\widetilde{Y}_1^k, \ldots, \widetilde{Y}_L^k\right)$ that can be easily generated (e.g. elliptical r.v.'s) written $\mathcal{L}_{\ell,c}^{[k,i_k]} = f_\ell^k\left(\widetilde{Y}_1^k, \ldots, \widetilde{Y}_L^k\right) - \mathbb{E}\left[f_\ell^k\left(\widetilde{Y}_1^k, \ldots, \widetilde{Y}_L^k\right)\right]$.
- For $\mathbf{Y} := \left\{Y_\ell^k\right\}_{\substack{1\le\ell\le L\\ 1\le k\le K}} \in \mathbb{R}^{LK}$, we write $\mathbf{Y} \bigoplus_{\mathbf{i}} \mathbf{Y}_0$ the updated version of $\mathbf{Y}$ where the components $Y_0^1, \ldots, Y_0^K$ are added to $Y_{i_1}^1, \ldots, Y_{i_K}^K$ resp.
- With $g_\ell(\cdot) = \sum_{k=1}^K f_\ell^k(\cdot) - \sum_{k=1}^K \mathbb{E}\left[f_\ell^k(\cdot)\right]$, $\mathbf{i} = (i_1, \ldots, i_K)$, $\mathcal{I} = \{1, \ldots, L\}^K$ the minimization of the **true problem** is

$$\min_{\mathbf{i}\in\{1,\ldots,L\}^K} \underbrace{\sum_{\ell=1}^L \mathcal{R}_\ell\left(g_\ell\left(\mathbf{Y}\bigoplus_{\mathbf{i}}\mathbf{Y}_0\right)\right)}_{=:H(\mathbf{i})}, \text{ and } \mathcal{I}^{opt} := \arg\min_{\mathbf{i}\in\mathcal{I}} \sum_{\ell=1}^L H(\mathbf{i}).$$

- In our problem, no access to $H(\mathbf{i})$, only to an approximated version with M samples. Let $\mathcal{R}_\ell^M$ the empirical version of $\mathcal{R}_\ell$ for each member ℓ based on $\mathbf{Y}^m$ and $\mathbf{Y}_0^m$, simulations of $\mathbf{Y}$ and $\mathbf{Y}_0$ resp., $m = 1, \ldots, M$. **Approximated problem** based on an approximated Hamiltonian to solve is

$$\min_{\mathbf{i}\in\{1,\ldots,K\}^L} \underbrace{\sum_{\ell=1}^L \mathcal{R}_\ell^M\left[g_\ell\left(\mathbf{Y}\bigoplus_{\mathbf{i}}\mathbf{Y}_0\right)\right]}_{:=H_M(\mathbf{i})}, \text{ and } \mathcal{I}_M^{opt} = \arg\min_{\mathbf{i}\in\mathcal{I}} H_M(\mathbf{i}).$$

Theoretical aspects of Combinatorial SA - known results (1 / 2)

For a given energy function H, consider the minimization problem of finding $\mathbf{i}^* \in \arg\min_{\mathbf{i}\in\mathcal{I}} H(\mathbf{i})$ with $H : \mathcal{I} \longrightarrow \mathbb{R}$. The SA algorithm runs for some predefined N iterations and approximates $\mathbf{i}^*$ by $\mathbf{i}^N$. Starting from $\mathbf{I}_{n-1} \in \mathcal{I}$, the state $\mathbf{I}'$ is suggested at time n, in some neighbourhood $\mathcal{O}(\mathbf{I}_{n-1})$ of $\mathbf{I}_{n-1}$ with a generation probability distribution given by $(P_n(\mathbf{i},\mathbf{j}))_{\mathbf{j}\in\mathcal{O}(\mathbf{i})}$ with $(P_n(\mathbf{i},\mathbf{j}))_{\mathbf{i},\mathbf{j}\in\mathcal{I}}$ a stochastic matrix.

The next iteration of the process $\mathbf{I}$ is $\mathbf{I}_n = \mathbf{I}'\mathbb{1}_{\{U_n \leq A_n(\mathbf{I}_{n-1},\mathbf{I}')\}} + \mathbf{I}_{n-1}\mathbb{1}_{\{U_n > A_n(\mathbf{I}_{n-1},\mathbf{I}')\}}$.

By (Delmas and Jourdain, 2006, Proposition 2.2.1, p. 35), $\mathbf{I}_n$ follows the discrete distribution whose probability coefficients are

$$Q_n(\mathbf{i},\mathbf{j}) = \begin{cases} P_n(\mathbf{i},\mathbf{j})A_n(\mathbf{i},\mathbf{j}) & \text{if } \mathbf{j} \neq \mathbf{i} \\ 1 - \sum_{\mathbf{j}'\neq\mathbf{i}} Q_n(\mathbf{i},\mathbf{j}') & \text{if } \mathbf{j} = \mathbf{i}. \end{cases}$$

Theorem 1

(Aarts and Korst, 1989, Theorem 3.3, page 42) Consider $H(\cdot)$ to minimize on $\mathcal{I}$ with a cooling schedule $(c_n, n \geq 1)$ s.t. $\lim_{n\to\infty} c_n = 0$. Assume that for any $k \geq 1$ the acceptance probabilities $(A(\mathbf{i},\mathbf{j},c))_{\mathbf{i},\mathbf{j}\in\mathcal{I}}$ for any temperature schedule value $c > 0$ and the generation probabilities $\left(P_k(\mathbf{i},\mathbf{j})\right)_{\mathbf{i},\mathbf{j}\in\mathcal{I}}$ are linked through $(Q_n(\mathbf{i},\mathbf{j}))_{\mathbf{i},\mathbf{j}\in\mathcal{I}}$ and respectively satisfies

1. *the generation transition probability distribution $\left\{\left(P_k(\mathbf{i},\mathbf{j})_{\mathbf{i},\mathbf{j}\in\mathcal{I}}\right)\right\}_{k\geq 1}$ is irreducible and symmetric*
2. $\forall k \geq 1,\ \forall \mathbf{i},\mathbf{j} \in \mathcal{I},\ \begin{cases} A(\mathbf{i},\mathbf{j},c) = 1 & if \quad H(\mathbf{i}) \geq H(\mathbf{j}) \\ A(\mathbf{i},\mathbf{j},c) \in (0,1) & if \quad H(\mathbf{i}) < H(\mathbf{j}) \end{cases}$
3. $\forall k \geq 1,\ \forall \mathbf{i},\mathbf{j},\mathbf{l} \in \mathcal{I}$ *with* $H(\mathbf{i}) \leq H(\mathbf{j}) \leq H(\mathbf{l})$, $A(\mathbf{i},\mathbf{l},c) = A(\mathbf{i},\mathbf{j},c)A(\mathbf{j},\mathbf{l},c)$
4. $\forall \mathbf{i},\mathbf{j} \in \mathcal{I}$ *with* $H(\mathbf{i}) < H(\mathbf{j})$: $\lim_{k\to\infty} A(\mathbf{i},\mathbf{j},c_k) = 0$.

Then there exists an invariant distribution π_n to the SA algorithm whose components, for arbitrary $\mathbf{i}_0 \in \mathcal{I}$, are given by

$$\pi_n(\mathbf{i}) = \frac{A(\mathbf{i}_0,\mathbf{i},c_n)}{\sum_{\mathbf{j}\in\mathcal{I}} A(\mathbf{i}_0,\mathbf{j},c_n)}, \quad \textit{for all } \mathbf{i} \in \mathcal{I} \quad \textit{and} \quad \lim_{n\to\infty} \pi_n(\mathbf{i}) = \frac{1}{|\mathcal{I}^{opt}|}\mathbb{1}_{\{\mathbf{i}\in\mathcal{I}^{opt}\}}.$$

Theoretical aspects of Combinatorial SA - known results (2 / 2)

Let $\mathbf{i}$ s.t. $H(\mathbf{i}) > \mathcal{H} = \min_{\mathbf{j} \in \mathcal{I}} H(\mathbf{j})$, $\gamma = (\mathbf{i}_0, \ldots, \mathbf{i}_n)$ a possible trajectory from $\mathbf{i}$ to $\mathbf{i}^* \in \mathcal{I}^{opt}$ with $\mathbf{i}_0 = \mathbf{i}$ and $\mathbf{i}_n = \mathbf{i}^*$ i.e. $\mathbf{i}_n \in \mathcal{I}^{opt}$ and $P(\mathbf{i}_l, \mathbf{i}_{l+1}) > 0$ for $0 \leq l < n$. $H(\gamma) := \max_{0 \leq l \leq n-1} H(\mathbf{i}_l) - \mathcal{H}$. Let $\Gamma_{\mathbf{i}}$ be the set of all possible trajectories from $\mathbf{i}$ to $\mathcal{I}^{opt}$ and $\underline{H}_{\mathbf{i}} = \min_{\gamma \in \Gamma_{\mathbf{i}}} H(\gamma)$. $\overline{H} = \max_{\mathbf{i} \in \mathcal{I}} \underline{H}_{\mathbf{i}}$ depends on both $H(\cdot)$ and $\Big(P(\mathbf{i}, \mathbf{j})\Big)_{\mathbf{i}, \mathbf{j} \in \mathcal{I}}$. $\underline{H} = \min_{\mathbf{i} \in \mathcal{I}} \underline{H}_{\mathbf{i}}$. Define $D = \max\limits_{\mathbf{i}: H(\mathbf{i}) > \mathcal{H}} (\underline{H}_{\mathbf{i}})/(H(\mathbf{i}) - \mathcal{H}) - 1 \in [0, (\overline{H}/\underline{H}) - 1]$ be the difficulty associated to $H(\cdot)$.

Theorem 2

Hajek (1988, Theorem 1) We have $\lim_{n \to \infty} \mathbb{Q} H(\mathbf{I}_n) > \mathcal{H} = 0$ *iff* $\lim_{n \to \infty} c_n = 0$ *and* $\sum_{n=1}^{\infty} e^{-H^*/c_n} = \infty$.

Theorem 3

Delmas and Jourdain (2006, Theorem 2.3.9) There are two constants $B_2 \geq B_1 \geq 0$ *such that for all* $N \geq 1$, $\frac{B_1}{N^{1/D}} \leq \max_{\mathbf{i} \in \mathcal{I}} \inf_{c_0 \geq \cdots \geq c_N} \mathbb{P}_{(c_0, \ldots, c_N)} \Big(H(\mathbf{I}_N) > \mathcal{H} \,\big|\, \mathbf{I}_0 = \mathbf{i}\Big) \leq \frac{B_2}{N^{1/D}}$, *where* $\mathbb{P}_{(c_0, \ldots, c_N)}$ *is the probability measure under the considered cooling schedule* $c_0 \geq \cdots \geq c_N$. *And for all* $A > 0$, *there exists* $\delta_A > 0$ *s.t. for all* N, *the cooling schedule* $\Big(c_0^{(N,A)}, \ldots, c_N^{(N,A)}\Big)$ *with* $c_n^{(N,A)} = \frac{1}{A}\left(\frac{A}{\log(N)^2}\right)^{\frac{n}{N}}$, *satisfies*

$$\max_{\mathbf{i} \in \mathcal{I}} \mathbb{P}\Big(H(\mathbf{I}_{N,A}) > \mathcal{H} \,\big|\, \mathbf{I}_0 = \mathbf{i}\Big) \leq \delta_A \left(\frac{\log(N) \log\Big(\log(N)\Big)}{N}\right)^{1/D},$$

where $\mathbf{I}_{N,A}$ *is the algorithm proposed solution after* N *iterations based on the cooling schedule* $\Big(c_0^{(N,A)}, \ldots, c_N^{(N,A)}\Big)$.

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Theoretical aspects of Combinatorial SA - error bounds (1 / 2)

Consider $H_M(\cdot)$ defined on $\mathcal{I}$ that depends on M Monte-Carlo samples $\mathbf{X}_{1:M} = (X_1, \ldots, X_M)$ (e.g. $X_m = g_\ell\left(\mathbf{Y}^m \bigoplus_{\mathbf{i}} \mathbf{Y}_0^m\right)$, $m \in 1 \ldots M$). Denote by $B : \mathbb{R}_+^* \times \mathbb{N}^* \to \mathbb{R}_+^*$, the function s.t., for any $\mathbf{i} \in \mathcal{I}$, for any $\varepsilon > 0$, for any $M \in \mathbb{N}^*$, (Reiss, 1989; Boucheron et al., 2013; Chamakh et al., 2021)

$$\mathbb{P}\left(\sqrt{M}|H_M(\mathbf{i}) - H(\mathbf{i})| > \varepsilon\right) \leq B(\varepsilon, M).$$

Let A chosen independent from $\mathbf{X}_{1:M}$ and $D_{\mathbf{X}_{1:M}}$ be the difficulty associated to H_M. From Theorem 3, there exists $\delta^A_{\mathbf{X}_{1:M}} > 0$ such that for all N, using $c_n^{(N,A)} = \frac{1}{A}\left(\frac{A}{\log(N)^2}\right)^{\frac{n}{N}}$, satisfies $\mathbb{P}_{\mathbf{X}_{1:M}}\left(H_M\left(\mathbf{I}_M^N\right) > \mathcal{H}_M\right) \leq \max_{\mathbf{i}\in\mathcal{I}} \mathbb{P}_{\mathbf{X}_{1:M}}\left(H_M\left(\mathbf{I}_M^N\right) > \mathcal{H}_M \middle| \mathbf{I}_0 = \mathbf{i}\right) \leq \delta^A_{\mathbf{X}_{1:M}}\left(\frac{\log(N)\log_2(N)}{N}\right)^{\frac{1}{D_{\mathbf{X}_{1:M}}}}$, with $\log_2(N) = \log\left(\log(N)\right)$, $\delta^A_{\mathbf{X}_{1:M}}$ and $D_{\mathbf{X}_{1:M}}$ explicitly depending on $\mathbf{X}_{1:M}$.

Theorem 4

Let $\mathbf{i}^* \in \arg\min_{\mathbf{i}\in\mathcal{I}} H(\mathbf{i})$, $\mathbf{I}_M^* \in \arg\min_{\mathbf{i}\in\mathcal{I}} H_M(\mathbf{i})$ and $\mathbf{I}_M^N$ be the solution found by the algorithm to minimize $H_M(\cdot)$ over $\mathcal{I}$ after N iterations with $H_M(\mathbf{i})$ an empirical estimation of $H(\mathbf{i})$, $\mathbf{i} \in \mathcal{I}$, based on M i.i.d. samples $X_1, \ldots, X_M$. For any $\varepsilon > 0$, we have

$$\begin{cases} \mathbb{P}\left(\sqrt{M}\left(H_M\left(\mathbf{I}_M^N\right) - H(\mathbf{i}^*)\right) < -\varepsilon\right) \leq |\mathcal{I}|B(\varepsilon, M), \\ \mathbb{P}\left(\sqrt{M}\left(H_M\left(\mathbf{I}_M^N\right) - H(\mathbf{i}^*)\right) > \varepsilon\right) \leq \mathbb{E}\left[\delta^A_{\mathbf{X}_{1:M}}\left(\frac{\log(N)\log\left(\log(N)\right)}{N}\right)^{\frac{1}{D_{\mathbf{X}_{1:M}}}}\right] + B\left(\frac{\varepsilon}{2}, M\right). \end{cases}$$

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Corollary 3.1

(Estimation error) *Let $\mathbf{i}^* \in \arg\min_{\mathbf{i}\in\mathcal{I}} H(\mathbf{i})$, $\mathbf{I}_M^* \in \arg\min_{\mathbf{i}\in\mathcal{I}} H_M(\mathbf{i})$ and $\mathbf{I}_M^N$ some random solution identified by the algorithm after N iterations. For any $\varepsilon > 0$,*

$$\mathbb{P}\left(\sqrt{M}\left(H\left(\mathbf{I}_M^N\right) - H(\mathbf{i}^*)\right) \geq \varepsilon\right)$$

$$\leq |\mathcal{I}|B\left(\frac{\varepsilon}{2}, M\right) + \mathbb{E}\left[\delta^A_{\mathbf{X}_{1:M}}\left(\frac{\log(N)\log\left(\log(N)\right)}{N}\right)^{\frac{1}{D_{\mathbf{X}_{1:M}}}}\right] + B\left(\frac{\varepsilon}{4}, M\right).$$

Proposition 1

Under the same assumptions of Theorem 4,

$$\mathbb{P}\left(I_M^N \notin \mathcal{I}_M^{opt}\right) \leq \mathbb{E}\left[\delta^A_{\mathbf{X}_{1:M}}\left(\frac{\log(N)\log\left(\log(N)\right)}{N}\right)^{\frac{1}{D_{\mathbf{X}_{1:M}}}}\right],$$

$$\mathbb{P}\left(I_M^N \notin \mathcal{I}^{opt}\right) \leq |\mathcal{I}|B\left(\frac{\varepsilon}{2}, M\right) + \mathbb{E}\left[\delta^A_{\mathbf{X}_{1:M}}\left(\frac{\log(N)\log\left(\log(N)\right)}{N}\right)^{\frac{1}{D_{\mathbf{X}_{1:M}}}}\right] + B\left(\frac{\varepsilon}{4}, M\right).$$

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Result: Return $\mathbf{I}_M^N$ and $H_M\left(\mathbf{I}_M^N\right)$

Initialise $\mathbf{I}_M^N \leftarrow \mathbf{I}$ based on i_k minimizing cost on each CCP k separately, $k = 1, \ldots, K$;

Evaluate $\mathcal{H}_M^N \leftarrow H_M\left(\mathbf{I}_M^N\right)$;

$n \leftarrow 1$;

while $n \leq N$ **do**

  $c_n \leftarrow c/\ln(n+1)$ $\left(\text{or } c_n \leftarrow \frac{1}{A}\left(A\Big/\log(N)^2\right)^{n/N}\right)$;

  $k \leftarrow 1$;

  for $k = 1$ *to* K **do**

    Draw i_k based on net-gross effect probabilities;

    or homogeneous probabilities for all members of CCP k;

  end

  Define $\mathbf{I}' \leftarrow (i_1, \ldots, i_K)$ and evaluate $H_M(\mathbf{I}')$ and $H_M(\mathbf{I})$;

  Calculate $A_n(\mathbf{I}, \mathbf{I}') = \exp\left\{-\left(H_k(\mathbf{I}') - H_k(\mathbf{I})\right)^+ \Big/ c_k\right\}$;

  Draw a uniform r.v. U_n;

  if $U_n \leq A_n(\mathbf{I}, \mathbf{I}')$ **then**

    update $\mathbf{I} \leftarrow \mathbf{I}'$;

  end

  Evaluate $H_M(\mathbf{I})$;

  if $H_M(\mathbf{I}) < H_M(\mathbf{I}_M^N)$ **then**

    $\mathbf{I}_M^N \leftarrow \mathbf{I}$ and $\mathcal{H}_M^N \leftarrow H_M(\mathbf{I})$;

  end

  $n \leftarrow n + 1$;

end

Algorithm 0: Combinatorial discrete SA for CCPs Default resolution

Numerical example (2 / 6) setup

- Consider 4 CCPs with 11 common members indexed from 0 to 10 among which member 10 has defaulted
- The portfolio drivers $(Y_\ell^k)_{\substack{0\le\ell\le 10\\ 1\le k\le 4}}$ are all (correlated) Student r.v.'s with degree of freedom 3. The credit latent variables $(X_\ell)_{1\le\ell\le L}$ are (correlated) standard Gaussian r.v.'s $\mathcal{N}(0,1)$ (see also Table 2)
- all surviving members are assumed to have a one year default intensity of 1%
- for convergence nuemrical illustration and confidence levels, 100 runs of the algorithm has been considered with N varying from 25 to 3000 iteration steps, with a step of 25 iterations
- A quantile at 99.9% is considered for the risk measure of all members with it upper confidence level as estimated energy levels (Meeker et al., 2017, Appendix G, p. 497)
- The parameters of the various costs calculations are summarized in Table 1:

One-period length for default	1 year
Liquidation period at default Δ_l	1 year
Portfolio variations correlation ρ_c's	30%
Credit factors correlation ρ_m's	20%
Quantile level used for clearing members EC calculation	99.9%
Number of Monte-Carlo simulation (for credit cost and EC computations)	100,000

Table 1: XVAs calculation configuration

- The (simplified) loss r.v. for member ℓ on CCP k is

$$\mathcal{L}_\ell^{[k]} = \frac{\mathbb{1}_{X_\ell < B_\ell}}{1-\gamma_\ell} \frac{1}{1+\sum_{\ell'=1}^{10} \mathbb{1}_{X_{\ell'} < B_{\ell'}}} \sum_{\ell'=1}^{10} \mathbb{1}_{X_{\ell'} \ge B_{\ell'}} Y_{\ell'}$$

Numerical example (3 / 6)

portfolio id	size	trend	volatility
p0 (cm0)	**−32.41**	**−3.24**	**20%**
p1 (cm1)	22.18	2.22	21%
p2 (cm2)	−15.17	−1.52	22%
p3 (cm3)	−10.38	−1.04	23%
p4 (cm4)	−7.1	−0.71	24%
p5 (cm5)	−4.86	−0.49	25%
p6 (cm6)	3.33	0.33	26%
p7 (cm7)	−2.28	−0.23	27%
p8 (cm8)	−1.56	−0.16	28%
p9 (cm9)	0.9	0.09	29%
p20 (cm10)	**47.36**	**4.74**	**30%**

(a) **CCP 1**

portfolio id	size	trend	volatility
p10 (cm0)	−60.54	−6.05	35%
p11 (cm1)	45.88	4.59	36%
p12 (cm2)	−34.77	−3.48	37%
p13 (cm3)	26.35	2.63	38%
p14 (cm4)	**19.97**	**2.00**	**39%**
p15 (cm5)	15.13	1.51	40%
p16 (cm6)	−11.47	−1.15	39%
p17 (cm7)	−8.69	−0.87	38%
p18 (cm8)	6.59	0.66	37%
p19 (cm9)	4.99	0.5	36%
p21 (cm10)	**−3.44**	**−0.34**	**30%**

(b) **CCP 2**

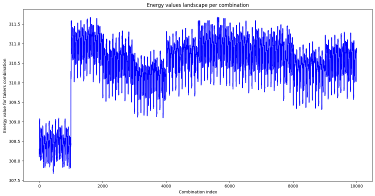
portfolio id	size	trend	volatility
p22 (cm0)	12.94	1.29	30%
p23 (cm1)	−11.27	−1.13	29%
p24 (cm2)	9.81	0.98	28%
p25 (cm3)	−8.54	−0.85	27%
p26 (cm4)	**7.43**	**0.74**	**26%**
p27 (cm5)	10.38	1.04	25%
p28 (cm6)	−25.89	−2.59	40%
p29 (cm7)	22.54	2.25	39%
p30 (cm8)	−19.62	−1.96	38%
p31 (cm10)	**17.08**	**1.71**	**37%**
p32 (cm9)	−14.87	−1.49	37%

(c) **CCP 3**

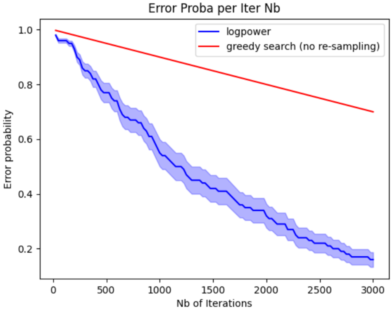
portfolio id	size	trend	volatility
p33 (cm2)	**−4.13**	**−0.41**	**20%**
p34 (cm1)	3.79	0.38	19%
p35 (cm0)	−3.47	−0.35	18%
p36 (cm3)	3.19	0.32	17%
p37 (cm4)	−2.92	−0.29	16%
p38 (cm5)	2.68	0.27	15%
p39 (cm10)	**−2.46**	**−0.25**	**21%**
p40 (cm8)	2.26	0.23	22%
p41 (cm7)	−2.07	−0.21	23%
p42 (cm6)	1.9	0.19	24%
p43 (cm9)	1.24	0.12	25%

(d) **CCP 4**

Table 2: CCPs and members portfolios with **defaulted member and corresponding portfolios**, **ground truth takers and their corresponding portfolios prior take-over**



(a) Energies landscape



(b) Error probabilities per N

Numerical example (5 / 6)

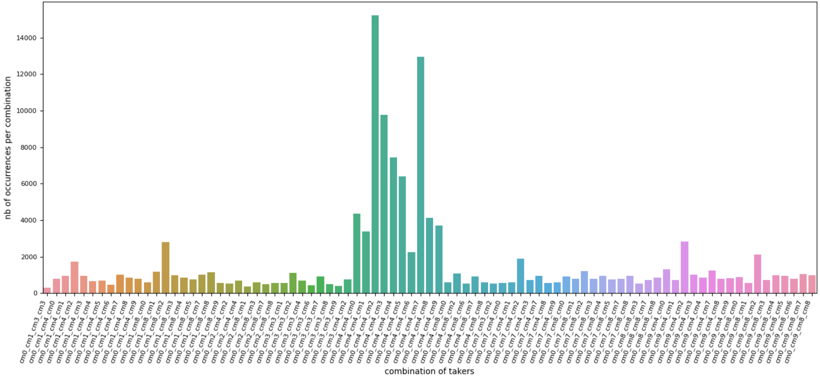


Figure 3: Combinations distribution for a number of occurence > 50.

Numerical example (6 / 6)

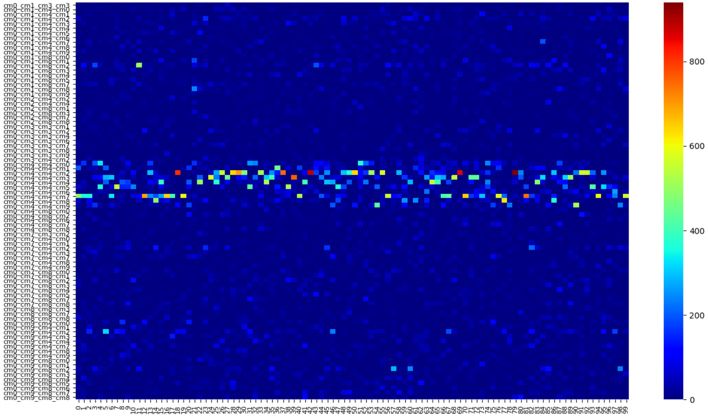


Figure 4: Combinations distribution for a number of occurrence > 50 per algorithm run. The band correspond to the top 6 combinations of the form $\mathrm{cm0\_cm4\_cm4\_cm}x$ with $x \in \{2, 3, 4, 5, 7, 8\}$.

Conclusion

- We reformulated an intricate financial network “deformation” problem as a combinatorial discrete SA optimization problem
- We emphasized the needed convergence results and error bounds when considering a certain type of cooling schedule
- We proposed a realistic example to test the approximated version of the algorithm
- We showed various encouraging results and analysis for applying such approximated algorithm

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Thank you!