A mutually exciting rough jump-diffusion (MERJD) for financial modelling

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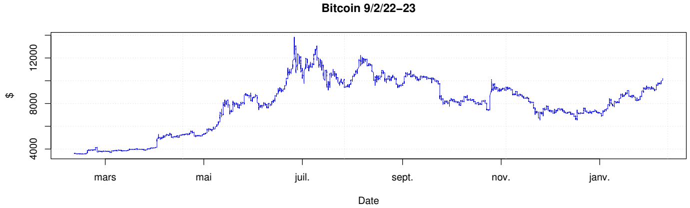
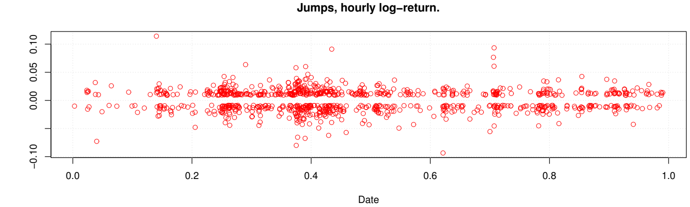
Introduction

The propensity of financial price jumps to cluster is a well known phenomenon. This is particularly true for crypto-currencies such as the Bitcoin.





Introduction

A natural way to replicate the clustering of jumps is offered by Hawkes self-exciting processes (Hawkes [3, 4]).

A self-exciting point processes, $L_t = \sum_{k=1}^{N_t} J_k$ has a jump arrival intensity λ_t ($i.e. \mathbb{E}(N_t|\mathcal{F}_{t-}) = \lambda_{t-} dt$) depending on the number of previous shocks.

The jump intensity increases after a shock and revert next to a baseline level, λ_0. The speed of reversion is determined by a memory kernel, $k(.)$.

$$\lambda_t = \lambda_0 + \int_0^t k(t-s)\, dN_s$$

We can draw a parallel between Hawkes processes and Brownian Volterra processes (BVP) of the form

$$X_t = X_0 + \int_0^t k(t-s) dW_s \,.$$

Introduction

Both are usually non-Markov and depend on a memory kernel $k(.)$. Among BVP, rough processes received a great deal of attention.

A rough process is close to a fractional Brownian motion (fBm) with a Hurst index $H < 1/2$.

At time $t > 0$, the rough fBm with $\alpha \in (1/2, 1]$ admits an integral representation with respect to a Bm, W_t :

$$\mathrm{fBm}_t \propto \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dW_s$$

This Brownian stochastic integral is well-defined even if the rough kernel $\frac{u^{\alpha-1}}{\Gamma(\alpha)}$, diverges when $u \to 0$.

This motivates us to study self-exciting processes with a dampened rough kernel:

$$\lambda_t = \lambda_0 + \eta \int_0^t e^{-\beta(t-s)} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dN_s \,.$$

MERJD dynamic

$(S_t)_{t\geq 0}$ is a price process and its log-return $X_t = \ln \frac{S_t}{S_0}$ is ruled by a mutually exciting rough jump diffusion (MERJD):

$$X_t = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t + \sum_{j=1}^{2} \left(L_t^{(j)} - \mu_j \int_0^t \lambda_s^{(j)} ds \right) ,$$

where $\left(L_t^{(1)}\right)_{t\geq 0}$ and $\left(L_t^{(2)}\right)_{t\geq 0}$ are positive and negative point processes,

$$L_t^{(j)} = \sum_{k=1}^{N_t^{(j)}} J_k^{(j)} \, , \, j = 1, 2 \, .$$

The distributions of $J_k^{(j)} \sim J^{(j)}$ for $j = 1, 2$, are $m^{(j)}(.)$ with $\mu_j = \mathbb{E}\left(J^{(j)}\right)$ and $\mathcal{J}_j(\omega) = \mathbb{E}\left(e^{\omega J^{(j)}}\right)$.

MERJD, dynamic

In examples, jumps are positive and negative expo. r.v. If $\rho_1 \in \mathbb{R}^+$ and $\rho_2 \in \mathbb{R}^-$. , the pdf's of $J^{(1)}$ and $J^{(2)}$ are

$$m^{(1)}(z) = \rho_1 e^{-\rho_1 z} 1_{\{z \geq 0\}} \, , \, m^{(2)}(z) = -\rho_2 e^{-\rho_2 z} 1_{\{z \leq 0\}} \, .$$

The intensities depend on pas $\left(N_t^{(j)}\right)_{t \geq 0}$ for $j = 1, 2$ in the following way:

$$\begin{pmatrix} \lambda_t^{(1)} \\ \lambda_t^{(2)} \end{pmatrix} = \begin{pmatrix} \lambda_0^{(1)} \\ \lambda_0^{(2)} \end{pmatrix} + \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \begin{pmatrix} \int_0^{t-} e^{-\beta(t-s)} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dN_s^{(1)} \\ \int_0^{t-} e^{-\beta(t-s)} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dN_s^{(2)} \end{pmatrix}$$

where $\alpha \in (0, 1]$, β , $\eta_{i,j} \in \mathbb{R}^+$ for $i, j \in \{1, 2\}$. We call $k(u) = e^{-\beta u} \frac{u^{\alpha-1}}{\Gamma(\alpha)}$, the dampened rough (or gamma) kernel.

The dampened rough kernel

The dampened rough kernel, belongs to the family of Sonine functions [6]:

Sonine function

A kernel $k(u) \in L^1_{loc}(\mathbb{R}^+)$ is a Sonine function if there exists a conjugate kernel $l(u) \in L^1_{loc}(\mathbb{R}^+)$ such that

$$\int_0^t l(t-u)\, k(u)\, du = 1\,,\ \forall t \geq 0. \tag{1}$$

Let $\phi \in L^1(\mathbb{R}^+)$, the Sonine operators associated to $k(u)$ and $l(u)$ are defined as

$$(K\phi)(t) = \int_0^t k\,(t-u)\,\phi\,(u)\, du\,,\ \forall t \geq 0, \tag{2}$$

$$(L\phi)(t) = \int_0^t l\,(t-u)\,\phi\,(u)\, du\,,\ \forall t \geq 0.$$

The dampened rough kernel

Given the similarity between $K\phi$ with $I^{\alpha}_{0+}\phi$, the Riemann-Liouville integral

$$K\phi = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{e^{-\beta(t-u)}\phi(u)}{(t-u)^{1-\alpha}} du \, , \quad I^{\alpha}_{0+}\phi := \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\phi(u)}{(t-u)^{1-\alpha}} du$$

we call the operator K as the dampened Riemann-Liouville (RL) integral. If $\phi \in L_1$ then $\alpha \in (0,1)$.

Proposition

The conjugate kernel $l(.)$ of $k(.)$ satisfying condition (1), is :

$$l(u) = \beta^{\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_u^{\infty} \frac{e^{-\beta s}}{s^{1+\alpha}} ds \, , \tag{3}$$

The dampened rough kernel

Proposition

The inverse operator of the dampened RL integral K, is the derivative of its conjugate kernel. For $\phi \in L^1(\mathbb{R}^+)$,

$$\begin{aligned}\left(K^{-1}\phi\right)(t) &= \frac{d}{dt}(L\phi)(t) \\ &= \frac{d}{dt}\int_0^t l(t-u)\,\phi(u)\,du\,.\end{aligned} \tag{4}$$

This inverse operator is called the dampened RL derivative.

Usefulness? The Laplace transform of $L_t^{(j)}$ will depend upon the solution of a fractional differential equation with operators K or K^{-1}.

MERJD, dynamics and first properties

Proposition

Let $u_j = \mathbb{E}\left(\lambda_t^{(j)}\right)$ for $j = 1, 2$. Expected intensities at time $t \geq 0$ conditionally to the filtration $\mathcal{F}_0$, are given by

$$\mathbb{E}_0 \begin{pmatrix} \lambda_t^{(1)} \\ \lambda_t^{(2)} \end{pmatrix} = \begin{pmatrix} \lambda_0^{(1)} \\ \lambda_0^{(2)} \end{pmatrix} + \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \begin{pmatrix} (Ku_1)(t) \\ (Ku_2)(t) \end{pmatrix}.$$

where $(Ku_j)(t) = \int_0^t k(t-u)\, u_j du$ are respectively equal to

$$(Ku_1)(t) = \int_0^t \left(\lambda_0^{(1)} + \eta_{12}(Ku_2)(s)\right) e^{-\beta(t-s)}(t-s)^{\alpha-1} E_{\alpha,\alpha}\left(\eta_{11}(t-s)^{\alpha}\right) ds$$
$$(Ku_2)(t) = \int_0^t \left(\lambda_0^{(2)} + \eta_{21}(Ku_1)(s)\right) e^{-\beta(t-s)}(t-s)^{\alpha-1} E_{\alpha,\alpha}\left(\eta_{22}(t-s)^{\alpha}\right) ds$$

Without cross contagion, we can integrate by parts Ku_1 and Ku_2.

MERJD, dynamics and first properties

Proposition

If the parameters defining $\lambda_t^{(1)}$ and $\lambda_t^{(2)}$ fulfill the following three conditions

$$\beta^\alpha \geq \eta_{11} \quad , \quad \beta^\alpha \geq \eta_{22} \, ,$$
$$(\beta^\alpha - \eta_{11})(\beta^\alpha - \eta_{22}) \geq \eta_{12}\eta_{21} \, ,$$

the expected intensities admit a limit when $t \to \infty$, that are:

$$\begin{pmatrix} \lambda_\infty^{(1)} \\ \lambda_\infty^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{\lambda_0^{(1)}(\beta^\alpha - \eta_{22})\beta^\alpha + \eta_{12}\lambda_0^{(2)}\beta^\alpha}{(\beta^\alpha - \eta_{11})(\beta^\alpha - \eta_{22}) - \eta_{12}\eta_{21}} \\ \frac{\lambda_0^{(2)}(\beta^\alpha - \eta_{11})\beta^\alpha + \eta_{21}\lambda_0^{(1)}\beta^\alpha}{(\beta^\alpha - \eta_{11})(\beta^\alpha - \eta_{22}) - \eta_{12}\eta_{21}} \end{pmatrix} .$$

MERJD, dynamics and first properties

Markov representation

For $j = 1, 2$, let us consider a family of auxiliary jump processes $Z_t^{(j,\xi)}$, indexed by $\xi \in \mathbb{R}^+$ with

$$dZ_t^{(j,\xi)} = -(\beta + \xi) Z_t^{(j,\xi)} dt + dN_t^{(j)} .$$

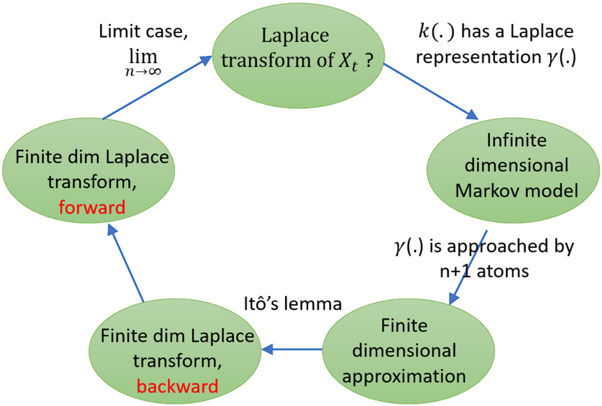
Le us denote $\gamma(d\xi) := \frac{\xi^{-\alpha}}{\Gamma(1-\alpha)} d\xi$ for $\xi \geq 0$. The intensities $\lambda_t^{(j)}$ are expressed as integrals of $Z_t^{(j,\xi)}$ with respect to $\gamma(d\xi)$:

$$\begin{pmatrix} \lambda_t^{(1)} \\ \lambda_t^{(2)} \end{pmatrix} = \begin{pmatrix} \lambda_0^{(1)} \\ \lambda_0^{(2)} \end{pmatrix} + \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \begin{pmatrix} \frac{1}{\Gamma(\alpha)} \int_0^\infty Z_t^{(1,\xi)} \gamma(d\xi) \\ \frac{1}{\Gamma(\alpha)} \int_0^\infty Z_t^{(2,\xi)} \gamma(d\xi) \end{pmatrix}$$

By construction, $\left(\lambda_t^{(1)}, \lambda_t^{(2)}, L_t^{(1)}, L_t^{(2)}, \left(Z_t^{(j,\xi)}\right)_{j \in \{1,2\}, \xi \in \mathbb{R}^+}\right)$ is Markov (inf. dim.).

MERJD, dynamics and first properties

I we find the Laplace transform of X_t, we can invert it numerically to retrieve the pdf. We adopt the following framework:



Laplace transform of the MERJD

Proposition

Laplace transform of the log-return $(X_s)_{s\geq 0}$, conditionally to $\mathcal{F}_0$:

$$\mathbb{E}\left(e^{-\omega X_s} \,|\, \mathcal{F}_0\right) = \exp\left(-\left(\omega\left(\mu - \frac{\sigma^2}{2}\right) - \frac{\omega^2\sigma^2}{2}\right) s + \boldsymbol{q}_\lambda(s)^\top \boldsymbol{\lambda}_0\right),$$

where $q_\lambda^{(j)}(s)$ for $j = 1, 2$ solves a forward ODE:

$$\frac{dq_\lambda^{(j)}(s)}{ds} = \omega\mu_j + \mathcal{J}_j(-\omega)\exp\left(\boldsymbol{\eta}_{.,j}^\top \left(K\frac{d\boldsymbol{q}_\lambda}{ds}\right)(s)\right) - 1 \qquad (5)$$

where $K\frac{dq_\lambda^{(j)}}{ds}$ is the dampened RL integral of $\frac{dq_\lambda^{(j)}}{ds}$:

$$\left(K\frac{dq_\lambda^{(j)}}{ds}\right)(s) = \frac{1}{\Gamma(\alpha)}\int_0^s \frac{e^{-\beta(s-u)}}{(s-u)^{1-\alpha}}\frac{dq_\lambda^{(j)}(u)}{du}\,du.$$

Laplace transform of the MERJD

In practice, we solve numerically Equation (5). We divide $[0, s]$ in n subintervals $[s_k, s_{k+1}]$ of length Δ, for $k = 0, ..., n-1$.

We denote by $\boldsymbol{g}(k) := \left.\frac{d\boldsymbol{q}_\lambda(s)}{ds}\right|_{s=s_k}$, the differential of $\boldsymbol{q}_\lambda$ at time s_k approximated by :

$$g^{(j)}(k) = \omega\mu_j + \mathbb{E}\left(e^{-\omega J^{(j)}}\right) \exp\left(\frac{\boldsymbol{\eta}_{.,j}^\top}{\Gamma(\alpha)} \sum_{u=0}^{k-1} \frac{e^{-\beta(s_k - s_u)}}{(s_k - s_u)^{1-\alpha}} \boldsymbol{g}(u)\,\Delta\right) - 1\,.$$

Initial value $g^{(j)}(0) = \omega\mu_j + \mathbb{E}\left(e^{-\omega J^{(j)}}\right) - 1$.

We use this to compute the pdf of $(X_t)_{t\geq 0}$ by Discrete Fourier Transform (e.g. for option pricing).

Econometric estimation

To illustrate this article, we fit the MERJD to time-series of hourly Bitcoin returns from the 9/2/2018 to 9/2/2023, traded in USD on the platform Gemini.

The bitcoin is traded 24h/24h and the time interval between two successive observations is $\Delta = 1/8760$ year.

Jumps are not directly observable. For this reason, we adopt a peak-over-threshold approach.

The record of p log-returns, lag Δ, is $\{x_{1,}x_1, x_2, ..., x_p\}$, at times $\{s_0, s_1, ..., s_p\}$.

Econometric estimation

The thresholds $g(\alpha_1)$ and $g(\alpha_2)$ depend on confidence levels, α_1 and α_2. We fit a pure Gaussian process to time-series:

$$x_k \sim \mu_g \Delta + \sigma_g W_\Delta$$

If $\Phi(.)$ is the cdf of a standard normal, $g(\alpha_1)$, $g(\alpha_2)$ are α_1 and α_2 percentiles:

$$g(\alpha_i) = \mu_g \Delta + \sigma_g \sqrt{\Delta} \Phi^{-1}(\alpha_i)$$

for $i = 1, 2$. The times of the k^{th} jump of $L_t^{(1)}$ and $L_t^{(2)}$ are :

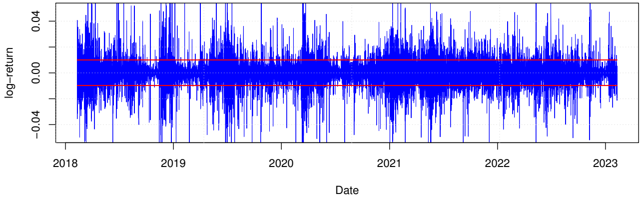
$$\begin{aligned} \tau_k^{(1)} &= \min\{s_j \in \{s_1, ..., s_p\} \,|\, x_j \geq g(\alpha_1)\,,\, s_j \geq \tau_{k-1}^{(1)}\}\,, \\ \tau_k^{(2)} &= \min\{s_j \in \{s_1, ..., s_p\} \,|\, x_j \leq g(\alpha_2)\,,\, s_j \geq \tau_{k-1}^{(2)}\}\,. \end{aligned}$$

The levels of confidence, α_1 and α_2, are optimized such that the skewness and kurtosis of x_i for periods without jumps are close to those of a normal distribution.

Econometric estimation

Skewness & kurtosis (hours without jumps): -9.19e-5 and 3.0002.

Parameters	Values	Parameters	Values
$g(\alpha_1)$	-0.9752%	$g(\alpha_2)$	1.0001%
$\widehat{\mu}\Delta$	0.0082%	$\widehat{\sigma}\sqrt{\Delta}$	0.3830%



$$\lambda_{t-}^{(j)} \quad = \quad \lambda_0^{(j)} + \sum_{k=1}^{2} \frac{\eta_{jk}}{\Gamma(\alpha)} \sum_{\tau_u^{(k)}<t} e^{-\beta(t-\tau_u^{(k)})}(t-\tau_u^{(k)})^{\alpha-1} \; j = 1, 2\,.$$

Econometric estimation

Parameters are obtained by log-likelihood maximization, which has an analytical expression:

Proposition

We denote the Gam. inc. function by $\Gamma(\alpha, x) = \int_x^\infty e^{-u} u^{\alpha-1} du$. The log-likelihood of observations is

$$\ln \mathcal{L} = \sum_{j=1}^{2} \left(-\int_0^{\mathcal{S}} \lambda_s^{(j)} ds + \sum_{k=1}^{N_{\mathcal{S}}^{(j)}} \log\left(\lambda_{\tau_k -}^{(j)}\right) \right),$$

where the integral of the intensity is equal to

$$\int_0^{\mathcal{S}} \lambda_s^{(j)} ds = \lambda_0^{(j)} \mathcal{S} + \sum_{k=1}^{2} \frac{\eta_{jk}}{\beta^\alpha} \sum_{u=1}^{N_{\mathcal{S}}^{(k)}} \left(1 - \frac{\Gamma\left(\alpha, \beta\left(\mathcal{S} - \tau_u^{(k)}\right)\right)}{\Gamma(\alpha)} \right).$$

Econometric estimation

After this, we estimate μ and σ from hourly observations, without jump. The jump size parameters are estimated by maximizing the log-likelihood of a mixed exponential-Gaussian distribution.

MERJD				Non-rough version			
$\widehat{\alpha}$	0.9061	$\widehat{\beta}$	181.7853	$\widehat{\alpha}$	1.0000	$\widehat{\beta}$	221.0378
$\widehat{\eta}_{11}$	48.6850	$\widehat{\eta}_{12}$	48.9050	$\widehat{\eta}_{11}$	107.089	$\widehat{\eta}_{12}$	90.60763
$\widehat{\eta}_{21}$	2.0365	$\widehat{\eta}_{22}$	87.1365	$\widehat{\eta}_{21}$	0.0416	$\widehat{\eta}_{22}$	174.0245
$\widehat{\lambda}_0^{(1)}$	53.8714	$\widehat{\lambda}_0^{(2)}$	101.8721	$\widehat{\lambda}_0^{(1)}$	38.0073	$\widehat{\lambda}_0^{(2)}$	106.9566
$\lambda_\infty^{(1)}$	488.9064	$\lambda_\infty^{(2)}$	505.8213	$\lambda_\infty^{(1)}$	473.9214	$\lambda_\infty^{(2)}$	503.2868
Log-lik. $N_t\ \mathcal{L}(\widehat{\Theta_N})$: 28 046.24				Log-lik. $N_t\ \mathcal{L}(\widehat{\Theta_N})$: 28 039.94			
$\widehat{\rho}_1$: 59.4260				$\widehat{\rho}_2$: -57.8427			

Significant difference? $2\left(\ln\mathcal{L}(\widehat{\Theta}_N) - \ln\mathcal{L}(\widehat{\Theta^h_N})\right) \sim \chi^2_1$ and p-value=0.0382%

Change of measure

The valuation of derivatives is performed under a risk-neutral measure $\mathbb{Q}$. Under $\mathbb{Q}$, discounted asset prices are martingales.

The market in our model, is incomplete. We focus on a family of changes of measure that are induced by exponential martingales of the form:

$$
\begin{aligned}
M_t &= \exp\left(-\frac{1}{2}\int_0^t \varphi(s)^2 ds - \int_0^t \varphi(s) dW_s\right) \times \\
&\quad \exp\left(\sum_{j=1}^{2}\left[\zeta_j L_t^{(j)} + (1-\mathcal{J}_j(\zeta_j))\int_0^t \lambda_s^{(j)} ds\right]\right),
\end{aligned}
$$

where $\varphi(t)$ is a $\mathcal{F}_t$-adapted process such that $\int_0^t |\varphi(s)|^2 ds < \infty$ and $\zeta_j \in \mathbb{R}$ are such that $\mathcal{J}_j(\zeta_j) = \mathbb{E}\left(e^{\zeta_j J^{(j)}}\right) < \infty$ for $j = 1, 2$.

Change of measure

Change of measure

For $j = 1, 2$, let us denote by $N_t^{Q(j)}$ the counting processes of intensity $\lambda_t^{Q(j)} = \mathcal{J}_j(\zeta_j)\lambda_t^{(j)}$. We define $J^{Q(j)}$, through their mgfs under the measure $\mathbb{Q}$:

$$\mathcal{J}_j^Q(\omega) = \mathbb{E}^{\mathbb{Q}}\left(e^{\omega J^{Q(j)}}\right) = \frac{\mathcal{J}_j(\omega + \zeta_j)}{\mathcal{J}_j(\zeta_j)}, \; j = 1, 2, \qquad (6)$$

and processes $L_t^{Q(j)} = \sum_{k=1}^{N_t^{Q(j)}} J_k^{Q(j)}$. Under the measure $\mathbb{Q}$,

$$\begin{aligned} X_t &= \left(\mu - \frac{\sigma^2}{2}\right) t - \sigma \int_0^t \varphi(s) ds + \sigma W_t^Q \\ &\quad + \sum_{j=1}^{2} \left(L_t^{Q(j)} - \frac{\mu_j}{\mathcal{J}_j(\zeta_j)} \int_0^t \lambda_s^{Q(j)} ds \right), \end{aligned} \qquad (7)$$

where $dW_t^Q = dW_t + \sigma\varphi(t)dt$.

Change of measure

Consequence: the equivalent measures $\mathbb{Q}$ defined by the change of measure are risk neutral if

$$\varphi(t) = \frac{\mu - r}{\sigma} + \sum_{j=1}^{2} \frac{\lambda_t^{(j)} \mathcal{J}_j(\zeta_j) \left(\mathbb{E}\left(e^{J^{Q(j)}} \right) - 1 \right) - \mu_j \int_0^t \lambda_s^{(j)} ds}{\sigma},$$

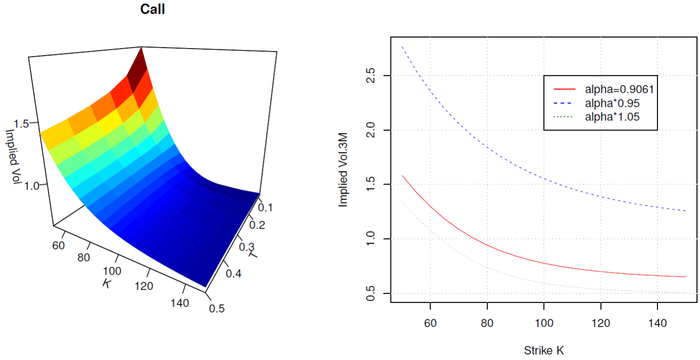
where r is the discount rate.

We evaluate European call options by DFFT and compute their implied volatility by inverting the Black & Scholes formula.

We use $\mathbb{P}$-parameters of bitcoin with $r = 0$ and set to zero the Brownian volatility to focus on the jump components of log-return.

Change of measure

$S_0 = 100$, strikes from 50 to 150 and expiry dates from 1 to 6 months



IV ranges from 61.39% up to 195.58%! Large but relevant with market data. The BitVol index (30-day implied volatility), evolves between 60% and 100% with a peak up to 168% on the 17/3/2020.

Conclusions

The rough self and mutually exciting processes are new types of non-Markov jump process, easy to combine with a diffusion.

Even if the memory kernel diverges at zero, the jump process remains stable under mild conditions. The MERJD admits an infinite dimensional Markov representation.

Considering the limit of a finite approximation allows us to retrieve the Laplace transform of the MERJD.

The DR kernel being a Sonine function, we can define a fractional operator close to the RL derivative. The Laplace transform of the MERJD depends on a solution of a fractional differential equation (FDE) using this new operator. This FDE can be solved numerically.

More info : A mutually exciting rough jump-diffusion for financial modelling, Frac. Calc. & Applied An. 27, 2024.

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