Multidimensional linear and nonlinear partial integro-differential equation in Bessel potential spaces with application in option pricing

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Partial integro-differential equations

- Examples of linear and nonlinear partial integro-differential equations
- Jump diffusion Lévy processes
- Partial integro-differential equations

Examples of PIDE in the first glance

Linear partial integro-differential equations

$$\frac{\partial u}{\partial \tau} = L[u], \quad u(0,x) = u^0(x), x \in \mathbb{R}^n, \tau \in (0,T)$$

$$\frac{\partial u}{\partial \tau} = \underbrace{\frac{\sigma^2}{2}\Delta u}_{\text{Differential part}} + \underbrace{\int_{\mathbb{R}^n} [u(x+z) - u(x) - z \cdot \nabla u]\, \nu(\mathrm{d}z)}_{\text{Nonlocal integral part}}$$

ν is the Lévy measure

Lévy process of jump-diffusion type

Representation of a Lévy stochastic jump diffusion process

$$X_t = \omega t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$$

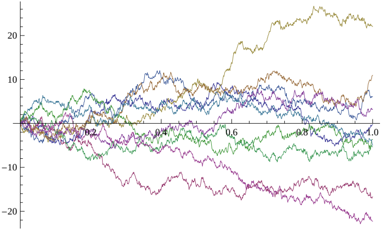
- $\sigma > 0$ is the volatility of the diffusion component, ω is a drift, $\{W_t, t \geq 0\}$, is a Wiener process
- N_t is a Poisson process with intensity λ counting jumps of X_t
- $Y_i, i = 1, 2, \cdots$, are i.i.d. random variables with density f
- the Lévy measure ν is given by λf, i.e. $\nu(dz) = \lambda f(z)dz$

- Finite activity process: $\nu(\mathbb{R}^n) = \int_{\mathbb{R}^n} \nu(dz) < \infty$
- Infinite activity process: $\nu(\mathbb{R}^n) = \int_{\mathbb{R}^n} \nu(dz) = \infty$

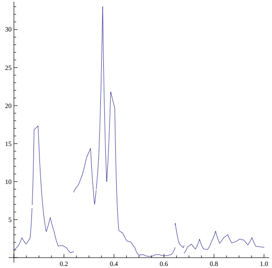
Examples of jump diffusion processes

- Merton's jump diffusion process
  $Y_i, i = 1, 2, \cdots$, are normally distributed with the Lévy density:

$$\nu(\mathrm{d}z) = \lambda \frac{1}{(2\pi\delta)^{\frac{n}{2}}} e^{-\frac{\|z-m\|^2}{2\delta^2}} \mathrm{d}z$$



Wiener process, $\omega = 0.5$, $\sigma = 2$



Merton's jump diffusion process,
$\omega = 0$, $\sigma = 5$, $m = 0$, $\delta = 1$, $\lambda = 20$

- Finite activity process: $\nu(\mathbb{R}) = \int_{\mathbb{R}} \nu(dx) = \lambda < \infty$
- Finite variation process: $\int_{|x|<1} x\nu(dx) < \lambda m < \infty$

Examples of jump diffusion processes $(n = 1)$

- The Variance Gamma process is a pure discontinuous process of infinite activity and finite variation with the Lévy measure:

$$\nu(\mathrm{d}x) = \frac{1}{\kappa|x|} e^{Ax - B|x|} \mathrm{d}x \quad \text{with} \quad A = \frac{\theta}{\sigma^2} \quad \text{and} \quad B = \frac{\sqrt{\theta^2 + 2\frac{\sigma^2}{\kappa}}}{\sigma^2}$$

- Normal Inverse Gaussian model is a process of infinite activity and infinite variation without any Brownian component with the Lévy measure: (K_ν is the Bessel function of the 2^{nd} kind)

$$\nu(\mathrm{d}x) = \frac{C}{|x|} e^{Ax} K_1\left(B\,|x|\right) \mathrm{d}x, \quad \text{where } B = \frac{\sqrt{\theta^2 + \frac{\sigma^2}{\kappa}}}{\sigma^2}, \quad C = \frac{\sqrt{\theta^2 + \frac{\sigma^2}{\kappa}}}{2\pi\sigma\sqrt{\kappa}}$$

PIDE derivation $(n = 1)$

- The Lévy measure of the process X_t is the mean number of jumps in X_t belonging to a Borel set A,

$$\nu(A) = \frac{1}{T}\mathbb{E}\left[J_X\left([0, T] \times A\right)\right], \quad J_X\left([0, t] \times A\right) = \#\left\{s \in [0, t] : \Delta X_s \in A\right\}$$

- X_t is a combination of a Brownian motion and Poisson processes

$$dX_t = \omega dt + \sigma dW_t + \int_{\mathbb{R}} x J_X\left(\mathrm{d}t, \mathrm{d}x\right), \quad X(0) = 0$$

Infinitesimal generator $L[u]$ of the associated semigroup

$$L[u](x) = \lim_{t \to 0^+} \frac{\mathbb{E}\left[u\left(x + X_t\right)\right] - u\left(x\right)}{t}$$

$$= \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + \int_{\mathbb{R}} \left[u\left(x + z\right) - u\left(x\right) - \left(e^z - 1\right)\frac{\partial u}{\partial x}\right] \nu(\mathrm{d}z),$$

Existence and uniqueness of solutions to PIDE

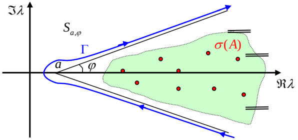
- Sectorial operators and analytic semigroups
- Bessel potential spaces and their representation
- Existence and uniqueness of solutions to PIDE in Bessel spaces

Theory of sectorial operators and analytic semigroups

Definition of a sectorial operator

A closed densely defined linear operator $A : D(A) \subset X \to X$ in a Banach space X is called sectorial iff there exists a sector $S_{a,\phi} = \{\lambda \in \mathbb{C},\ \phi \leq arg(\lambda - a) \leq 2\pi - \phi\}$ and $M > 0$ such that

$$\|(A-\lambda)^{-1}\| \leq M/|\lambda - a|, \quad \forall \lambda \in S_{a,\phi} \subset \mathbb{C} \setminus \sigma(A)$$



Spectrum $\sigma(A)$ of the operator A

$$e^{-At} = \frac{1}{2\pi i} \oint_{\Gamma} e^{-\lambda t}(\lambda - A)^{-1} d\lambda$$

$-A$ is a generator of an analytic semigroup $\{e^{-At}, t \geq 0\}$

- $e^{-At}e^{-As} = e^{-A(t+s)}$
- $\frac{d}{dt}e^{-At} = -Ae^{-At} \ \Rightarrow\ \frac{d}{dt}u + Au = 0,$ where $u(t) = e^{-At}u(0)$

$$a^{-\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{\gamma-1} e^{-at} dt \quad \Longrightarrow \quad A^{-\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{\gamma-1} e^{-At} dt$$

The fractional power space X^γ and fractional power operator A^γ

- $X^\gamma = D(A^\gamma) \equiv Rng(A^{-\gamma}) = \{u \in X,\ u = A^{-\gamma} v,\ v \in X\}$
- $\|u\|_\gamma = \|A^\gamma u\| = \|v\|,$ where $u = A^{-\gamma} v$
- $\|e^{-At}\|_\gamma \leq \frac{C}{t^\gamma} e^{-at}$ for any $t > 0$ and $\gamma \geq 0$

Henry's Theorem on existence and uniqueness of abstract equations

- $-A$ generates an analytic semigroup $\left\{e^{-At}, t \geq 0\right\}$ in X and the initial condition $U_0 \in X^\gamma$ where $0 \leq \gamma < 1$,
- $F : [0, T] \times X^\gamma \to X$ is Hölder continuous in τ and Lipschitz continuous mapping in U where $T > 0$.

Then, there exists a unique solution $U \in C([0, T], X^\gamma) \cap C^1((0, T), X)$

$$\frac{\partial U}{\partial \tau} + AU = F(\tau, U), \qquad U(0) = U_0.$$

Laplace operator as a generator of analytic semigroup

- The Laplace operator $-\Delta$ is sectorial in the Banach space $X = L^p(\mathbb{R}^n)$ of Lebesgue p-integrable functions for any $p \geq 1$
- The domain $D(A)$ is embedded into the Sobolev space $W^{2,p}(\mathbb{R}^n)$
- The fractional power space X^γ is the space of Bessel potentials:

$$X^\gamma = \mathscr{L}^p_{2\gamma}(\mathbb{R}^n) := \{G_{2\gamma} * \varphi,\ \varphi \in L^p(\mathbb{R}^n)\}$$

$$G_{2\gamma}(x) = \frac{(4\pi)^{-n/2}}{\Gamma(\gamma)} \int_0^\infty \xi^{-1+(2\gamma-n)/2} e^{-(\xi+\|x\|^2/(4\xi))} \mathrm{d}\xi$$

  where $G_{2\gamma}$ is the Bessel potential function with the Fourier transform $\hat{G}_{2\gamma}(y) = (1 + \|y\|^2)^{-\gamma}$
- The norm of $u = G_{2\gamma} * \varphi$ is given by $\|u\|_{X^\gamma} = \|\varphi\|_{L^p}$. The fractional power space X^γ is continuously embedded into the fractional Sobolev-Slobodeckii space $W^{2\gamma,p}(\mathbb{R}^n)$.

Henry, D. (1981). *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York.

Stein, E. M. (1970). *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J.

Existence and uniqueness of solutions to PIDE

Fundamental lemma (boundedness of the nonlocal part)

Assume $\nu(\mathrm{d}z)$ is an admissible activity Lévy measure, i.e.

$$\nu(\mathrm{d}z) = h(z)\mathrm{d}z, \qquad 0 \le h(z) \le C_0 \|z\|^{-\alpha} e^{-D\|z\|},$$

for all $z \in \mathbb{R}^n$ and the shape parameters $\alpha < n + 2, D > 0$. Suppose that $\gamma \geq 1/2$ and $\gamma > (\alpha - n)/2$. Then the mapping

$$f[u](x) = \int_{\mathbb{R}^n} [u(x+z) - u(x) - z \cdot \nabla_x u(x)] \, \nu(\mathrm{d}z)$$

is a bounded linear operator from X^γ into $X = L^p(\mathbb{R})$.

Existence and uniqueness of solutions to PIDE

The proof of the fundamental lemma is based on:

- representation of $u \in X^\gamma$ and $\nabla_x u \in X^{\gamma-1/2}$

  $$\nabla_x u(x+\theta z) - \nabla_x u(x) = [G_{2\gamma-1}(x+\theta z - \cdot) - G_{2\gamma-1}(x-\cdot)] * \varphi(\cdot).$$

  where $G_{2\gamma-1}$ is the Bessel potential function
- convolution inequality

  $$\|G * \varphi\|_{L^p} \le \|G\|_{L_q} \|\varphi\|_{L_r},$$

  where $p, q, r \ge 1$ and $1/p + 1 = 1/q + 1/r$
- and Hölder continuity of $G_{2\gamma-1}$

  $$\|G_{2\gamma-1}(\cdot + h) - G_{2\gamma-1}(\cdot)\|_{L_1} \le C_1 |h|^{2\gamma-1},$$

$$G_{2\gamma-1}(x) = \frac{(4\pi)^{-n/2}}{\Gamma(\gamma-1/2)} \int_0^\infty \xi^{-1+(2\gamma-1-n)/2} e^{-(\xi+\|x\|^2/(4\xi))} \mathrm{d}\xi$$

Stein, E. M. (1970). *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J.

Existence and uniqueness of solutions to PIDE

Applying Henry's Theorem on existence and uniqueness of abstract equations we conclude:

Theorem

Assume $\nu(\mathrm{d}z)$ *is an admissible activity Lévy measure with the shape parameters* $\alpha < n+2$. *Let* $X^\gamma = \mathscr{L}^p_{2\gamma}(\mathbb{R})$ *be the space of Bessel potentials, that is, the fractional power space of* $X = L^p(\mathbb{R}^n), p \geq 1$, *with respect to* $A = -\Delta$ *where* $\frac{1}{2} \leq \gamma < 1$ *and* $\frac{\alpha-n}{2} < \gamma < 1$.
Then, for any $T > 0$, *the linear PIDE*

$$\frac{\partial u}{\partial \tau} = \frac{\sigma^2}{2}\Delta u + \int_{\mathbb{R}^n} [u(x+z) - u(x) - z \cdot \nabla_x u(x)]\, \nu(\mathrm{d}z)$$

has the unique solution $u \in C([0,T], X^\gamma) \cap C^1((0,T), X)$ *satisfying a given initial condition* $u(0,\cdot) = u^0 \in X^\gamma$. *Moreover,* $u(\tau,\cdot) \in X^1 = \mathscr{L}^p_2(\mathbb{R}^n) \subseteq W^{2,p}(\mathbb{R}^n)$ *for any* $\tau \in (0,T)$.

Applications of PIDE in financial modelling

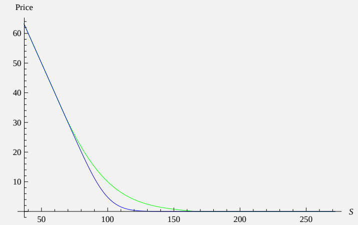
- Pricing derivative securities under jump diffusion processes

Application on pricing securities under Lévy process

In a stylized market the price $V(t, S)$ of a Call or Put option on the asset price S following the Lévy jump diffusion process is a solution to:

Black-Scholes Partial Integro-differential equation

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV + \int_{\mathbb{R}} \underbrace{\left[V(t, Se^y) - V(t, S) - (e^y - 1) S \frac{\partial V}{\partial S}(t, S) \right]}_{>0 \quad \text{if } S \mapsto V(S,t) \quad \text{is convex} \quad \Rightarrow \quad V^{PIDE}(S,t) > V^{BS}(S,t)} \nu(\mathrm{d}y) = 0,$$



$$V(T, S) = \max(K - S, 0), \quad S > 0$$

Application on pricing securities under Lévy process

The price $V(t, S_1, \cdots, S_n)$ of a Call or Put basket option on the asset prices $S_i, i = 1, \cdots, n$ following the Lévy jump diffusion process is a solution to:

Black-Scholes Partial Integro-differential equation

$$\begin{aligned}\frac{\partial V}{\partial t} &+ \frac{1}{2}\sum_{i,j=1}^{n} \varrho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + r\sum_{i=1}^{n} S_i \frac{\partial V}{\partial S_i} - rV \\ &+ \int_{\mathbb{R}^n} \Big[V(t, S_1 e^{y_1}, \cdots, S_n e^{y_n}) - V(t, S_1, \cdots, S_n) \\ &\qquad - \sum_{i=1}^{n} (e^{y_i} - 1) S_i \frac{\partial V}{\partial S_i}(t, S) \Big] \nu(\mathrm{d}y) = 0,\end{aligned}$$

Basket option terminal condition

$$V(T, S_1, \cdots, S_n) = \max(K - \sum_{i=1}^{n} S_i,\ 0), \quad S > 0$$

Thank you for your attention

The presentation is based on papers

- J. Cruz and D. Ševčovič: On solutions of a partial integro-differential Black-Scholes equation in Bessel potential spaces, Japan Journal of Industrial and Applied Mathematics 37 (2020), 691-721.
- J. Cruz and D. Ševčovič: Option Pricing in Illiquid Markets with Jumps, Applied Mathematical Finance, 25(4), 2018, 389-409.
- D. Ševčovič and C. Udeani: Multidimensional linear and nonlinear partial integro-differential equation in Bessel potential spaces with applications in option pricing, Mathematics 2021, 9(13), 1463.