Innovation and Product Positioning in a Monopoly

Anne Balter[1] Cláudia Nunes[2] Diogo Pereira[2] Peter Kort[1]

[1]Tilburg University, The Netherlands

[2]CEMAT and IST - University of Lisbon, Portugal

The problem

A firm is in the market, but plans to introduce a new product, deciding on:

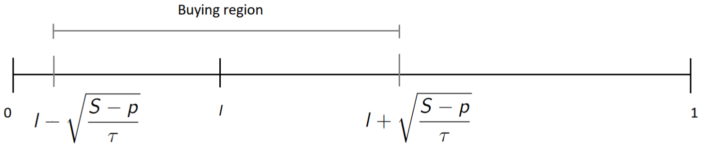
- When to introduce the new product;
- Where the new product will be sold (horizontal differentiation), in the Hotelling line [0,1];
- How much to invest in R&D in order to improve the new product (vertical differentiation);
- The prices of the products.

Characteristics of the old product (before the new one is introduced)

- The old product is located in 0, in the Hotelling line [0,1];
- Its price is constant and known, equal to p_O^O, with utility S^O;
- C^O is its cost of production;
- A consumer in location $x \in [0,1]$ buys the product if

$$S^O - p_O^O - \tau(x-l)^2 \geq 0$$

where $\tau > 0$ is the traveling cost.



Dynamics of the consumers

- Over time the number of consumer evolves randomly as a geometric Brownian motion

$$dY_t = \mu Y_t \, dt + \sigma Y_t \, dW_t.$$

- Consumers are uniformly distributed in the Hotelling line.

Characteristics of the new product

- Its production cost is C^N;
- If the firm invested R in R&D in order to improve the product then its utility is $S^N(R) \in C^2([0,\infty))$, with

$$\frac{dS^N}{dR} > 0, \qquad \text{and} \qquad \frac{d^2S^N}{dR^2} < 0.$$

- Its location is l^N;
- p_N^N denotes its price;
- p_O^N denotes the price of the old product (with the new product in the market).

Before the introduction of the new product:

Market share of the old product: $M^O(p_O^O)$, defined by the length of the set

$$\left\{x \in [0,1] \,\middle|\, S^O - p_O^O - \tau x^2 \geq 0\right\}$$

Market shares (cont.)

After the introduction of the new product:

Market share of the old product: $M_O^N(p_O^N, p_N^N, l^N, R)$, defined by the length of the set

$$\left\{ x \in [0,1] \,\middle|\, S^O - p_O^N - \tau x^2 \geq 0, \right.$$

$$\left. \text{and} \quad S^O - p_O^N - \tau x^2 \geq S^N(R) - p_N^N - \tau(x - l^N)^2 \right\}.$$

Market share of the new product: $M_N^N(p_O^N, p_N^N, l^N, R)$, defined by the length of the set

$$\left\{ x \in [0,1] \,\middle|\, S^N(R) - p_N^N - \tau(x - l^N)^2 \geq 0, \right.$$

$$\left. \text{and} \quad S^O - p_O^N - \tau x^2 \leq S^N(R) - p_N^N - \tau(x - l^N)^2 \right\}.$$

Optimization problem

Maximize in T, R, p_O^N, p_N^N and I^N

$$\mathbb{E}\Big[\int_0^T M_O^O(p_O^O)\,(p_O^O - C^O)\,Y_t\,e^{-\rho t}\,dt - (I+R)e^{-\rho T} + \int_T^\infty \Big(M_O^N(p_O^N, p_N^N, I^N, R)\,(p_O^N - C^O) + M_N^N(p_O^N, p_N^N, I^N, R)\,(p_N^N - C^N)\Big)\,Y_t\,e^{-\rho t}\,dt\Big]$$

where T is a stopping time and $I > 0$ is the investment cost.

Re-writing the problem

With

$$Q_O^* = M_O^O(p_O^O)(p_O^O - C^O)$$

and

$$J(y) = \max_{p_O^N, p_N^N, I^N, R} \Bigg(-(I + R) + \Big(M_O^N(p_O^N, p_N^N, I^N, R)\,(p_O^N - C^O) + M_N^N(p_O^N, p_N^N, I^N, R)\,(p_N^N - C^N) \Big) \frac{y}{\rho - \mu} \Bigg)$$

The problem is

$$\sup_T \mathbb{E}\left[\int_0^T Q_O^*\, Y_t\, e^{-\rho t}\, dt + J(Y_T)\, e^{-\rho T} \right]$$

First result: optimal prices and location

For fixed R:

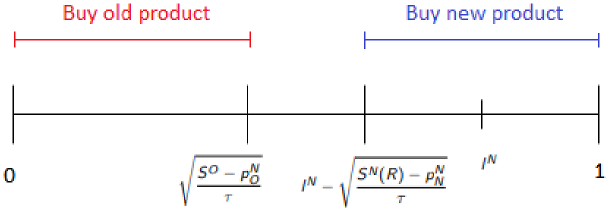
p_N^N	p_O^N	l^N	in case	strategy
$\frac{1}{3}(2\,S^N(R)+C^N)$	$\frac{1}{3}(2\,S^O+C^O)$	any	$2\sqrt{\frac{S^N(R)-C^N}{3\tau}}+\sqrt{\frac{S^O-C^O}{3\tau}}\leq 1$	add, non-full market
$S^N(R)-\tau(*)^2$	$S^O-\tau(1-2(*))^2$	$1-(*)$	$2\sqrt{\frac{S^N(R)-C^N}{3\tau}}+\sqrt{\frac{S^O-C^O}{3\tau}}>1$ $-3<\frac{S^N(R)-C^N}{\tau}-\frac{S^O-C^O}{\tau}<\frac{3}{4}$	add, full-market
$S^N(R)-\frac{1}{4}\tau$	–	$\frac{1}{2}$	$2\sqrt{\frac{S^N(R)-C^N}{3\tau}}+\sqrt{\frac{S^O-C^O}{3\tau}}>1$ $\frac{S^N(R)-C^N}{\tau}-\frac{S^O-C^O}{\tau}\geq\frac{3}{4}$	replace

where

$$(*)=\frac{2}{3}-\frac{1}{3}\sqrt{1-\frac{S^N(R)-C^N}{\tau}+\frac{S^O-C^O}{\tau}}.$$

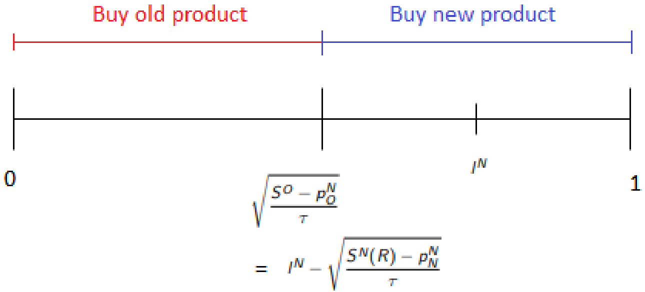
p_N^N	p_O^N	l^N	in case	strategy
$\frac{1}{3}(2\,S^N(R)+C^N)$	$\frac{1}{3}(2\,S^O+C^O)$	any	$2\sqrt{\frac{S^N(R)-C^N}{3\tau}}+\sqrt{\frac{S^O-C^O}{3\tau}}\leq 1$	add, non-full market

Add, non-full market :



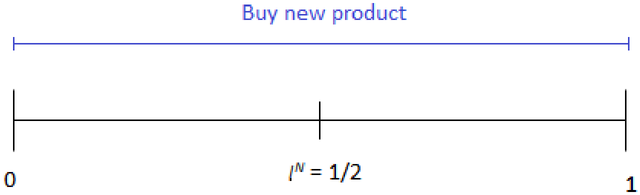
p_N^N	p_O^N	l^N	in case	strategy
$S^N(R) - \tau(*)^2$	$S^O - \tau(1-2(*))^2$	$1-(*)$	$2\sqrt{\frac{S^N(R)-C^N}{3\tau}} + \sqrt{\frac{S^O-C^O}{3\tau}} > 1$ $-3 < \frac{S^N(R)-C^N}{\tau} - \frac{S^O-C^O}{\tau} < \frac{3}{4}$	add, full-market

Add, full-market :



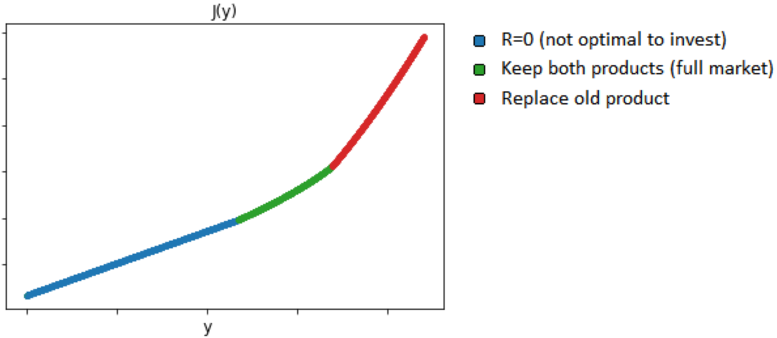
p_N^N	p_O^N	l^N	in case	strategy
$S^N(R) - \frac{1}{4}\tau$	$-$	$\frac{1}{2}$	$2\sqrt{\frac{S^N(R)-C^N}{3\tau}} + \sqrt{\frac{S^O-C^O}{3\tau}} > 1$ $\frac{S^N(R)-C^N}{\tau} - \frac{S^O-C^O}{\tau} \geq \frac{3}{4}$	replace

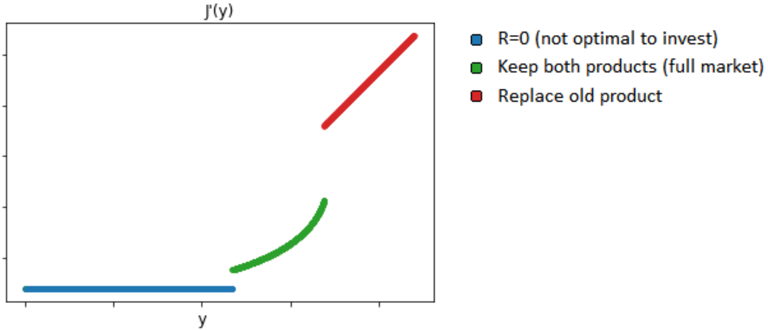
Replace :



The case $S^N(R) = \sqrt{R}$

In this case the function $J(y)$ can be found explicitly, it is continuous in all its domain but in general is not differentiable.




J'(y)
y
R=0 (not optimal to invest)
Keep both products (full market)
Replace old product

There are 5 possible cases for J, and they depend on the following conditions (all depending on exogeneous parameters):

$$C^N \leq > \frac{1}{4}\tau\left(1-\sqrt{\frac{S^O-C^O}{3\,\tau}}\right)^2 \quad \text{and} \quad C^N \leq > \frac{3}{4}\tau-(S^O-C^O)$$

$$\text{and} \quad C^N \leq > 3\tau-(S^O-C^O)$$

HJB equation

The usual approach is to solve the HJB equation

$$\max\left\{ \mu y V'(y) + \frac{1}{2}\sigma^2 y^2 V''(y) - \rho V(y) + y Q_O^* \ , \ J(y) - V(y) \right\} = 0$$

with the optimal stopping time given by

$$T^* = \inf\left\{ t \geq 0 \ \middle| \ V(Y_t) = J(Y_t) \right\}$$

Solving the HJB equation is usually done by "guessing" the solution. In particular, by guessing the area where one invests/waits, such that the constructed V is C^1.

Explicit expectation

For the stopping time

$$T_y = \inf \left\{ t \geq 0 \,\middle|\, Y_t \geq y \right\}$$

The expectation that defines our objective can be solved explicitly:

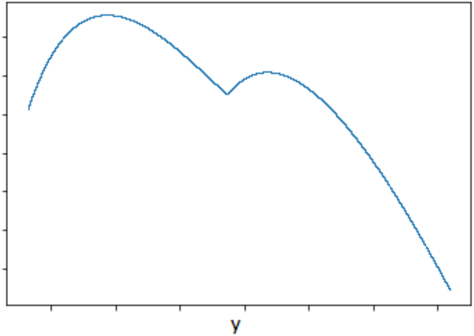
$$E\left[\int_0^{T_y} Q_O^* \, Y_t \, e^{-\rho t} \, dt + J(Y_{T_y}) \, e^{-\rho \, T_y}\right]$$
$$= Q_0^* \frac{Y_0}{\rho - \mu} + \left(J(y) - \frac{Q_O^*}{\rho - \mu} y\right) \, y^{-\beta}$$

and we want to find y that maximizes this expression. Note that

$$\left(J(y) - \frac{Q_O^*}{\rho - \mu} y\right) \, y^{-\beta}, \tag{1}$$

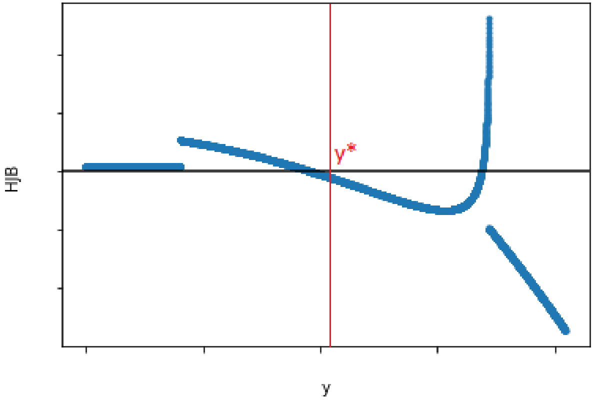
does not depend on Y_0.

For some parameter cases, $\left(J(y) - \frac{Q_O^*}{\rho-\mu} y\right) y^{-\beta}$ has a plot of the form (2 local maximums)



But this contradicts some T_y being optimal for all Y_0.

If plotting $\mu y J'(y) + \frac{1}{2}\sigma^2 y^2 J''(y) - \rho J(y) + y Q_O^*$ we have



where y^* is the maximum of $\left(J(y) - \frac{Q_O^*}{\rho-\mu} y\right) y^{-\beta}$.
To solve the HJB, this plot must be negative for $y \geq y^*$, which does not happen here.

Two stopping regions

- In this sense, this problem requires possibly two stopping regions (hysteresis);
- We have also found that it is never optimal to invest for $y \geq y^*$ if y^* is a point where J is not differentiable:

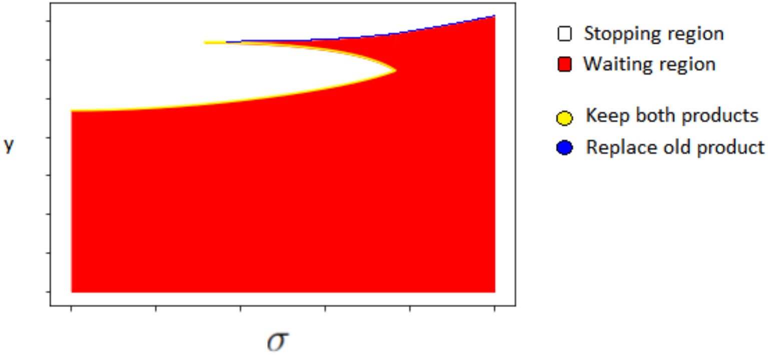
$$\lim_{\varepsilon \to 0^+} \frac{J(y^* + \varepsilon) - J(y^*)}{\varepsilon} > \lim_{\varepsilon \to 0^-} \frac{J(y^* + \varepsilon) - J(y^*)}{\varepsilon}.$$

Results

It is then possible to find the solution to the problem, which means that given a set of parameters (production prices, drift and volatility of the demand, travelling costs, utility function, investment cost), we find the optimal rule for the investment (meaning time, location, prices and how much to invest in R&D).

Decision as a function of σ

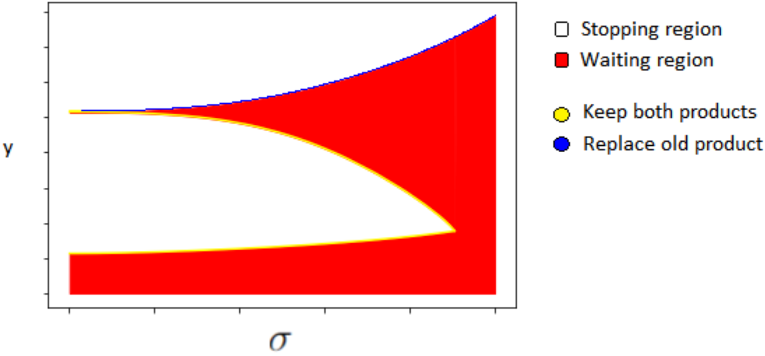
Example 1: J is differentiable



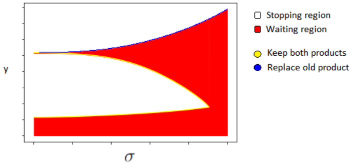
(even with J being differentiable, you may have a hysteresis region)

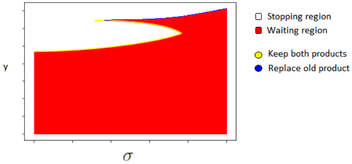
Decision as a function of σ

Example 2: J is not differentiable



In both cases:

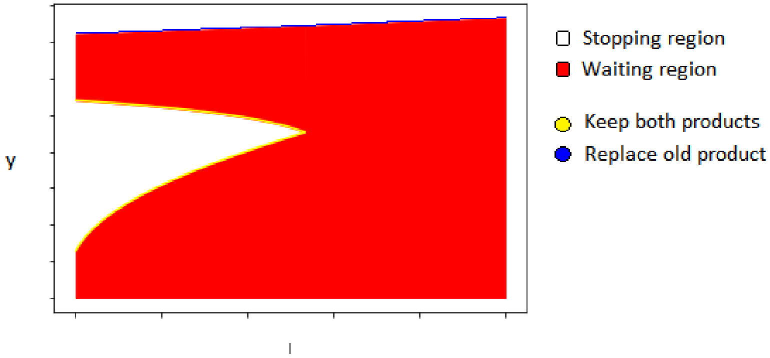




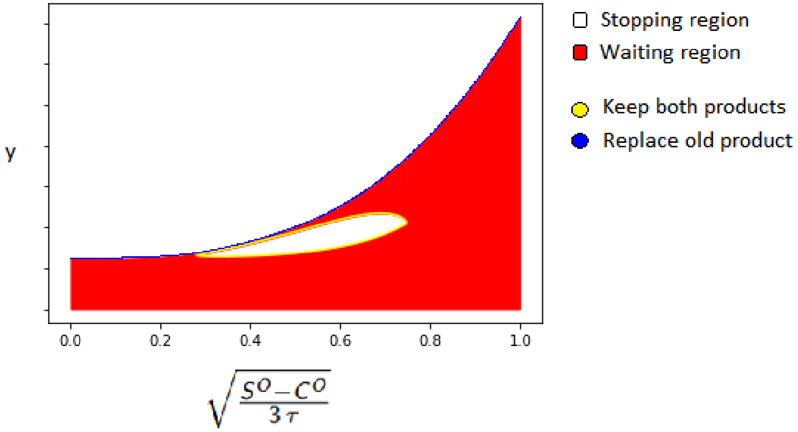
- The optimal investment threshold and the *R&D* are related

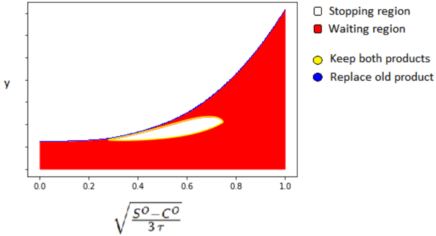
$$y^* = (\rho - \mu) \frac{I + R}{Q_N^*(R) - Q_O^*} \frac{\beta}{\beta - 1},$$

  so a higher investment threshold means a higher *R&D*.
- High uncertainty (σ) leads to late investment (with higher *R&D*.)
- For sufficiently large values of uncertainty, the old product is abolished and replaced by the new product.


y
I
Stopping region
Waiting region
Keep both products
Replace old product

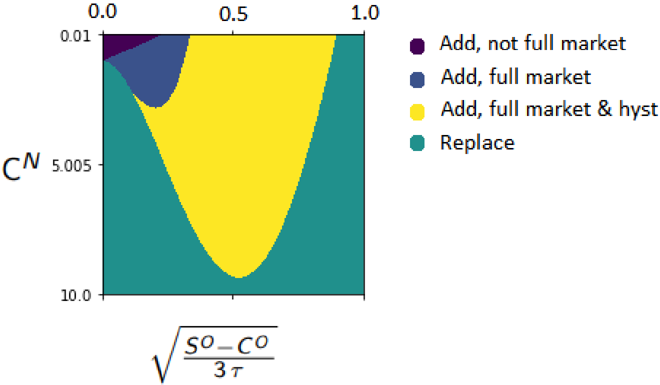
Decision as a function of the previous market share



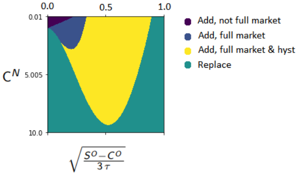


- If the old product is of good quality (large S^0), then one needs a good new product for it to be worth investing.
- The quality required can be very high, making it only worth investing by replacing the old product.
- For equally profitable products we keep both products.

In terms of C^N and $\sqrt{\frac{S^O - C^O}{3\tau}}$, the optimal solution is ...



Here $\sqrt{\frac{S^O - C^O}{3\tau}}$ is the optimal market share of the old product (without the new product)



- If the cost of the new product C^N is large enough, it requires a large $R\&D$ and the firm invests late. In this case it will be optimal to replace the old product (green zone)
- If the old product is profitable (high $\sqrt{\frac{S^O - C^O}{3\tau}}$), then one needs a good new product. It will also be optimal to replace.

Conclusions

- If the new product being introduced is similar to the old one, we usually want to keep both products.
- Higher volatility or investment costs means we wait longer for better market conditions.
- High costs of production for the new product means we wait longer and replace the old product.

Future extension

Consider duopoly, leading to competition.

Thank you!