Efficient Fourier Pricing of Multi-Asset Options: Quasi-Monte Carlo & Domain Transformation Approach

Chiheb Ben Hammouda

Utrecht University

Christian Bayer
Weierstrass Institute for Applied Analysis and Stochastics

Michael Samet
UQ RWTH Aachen University

Antonis Papapantoleon
TU Delft

Raúl Tempone
UQ RWTH Aachen University

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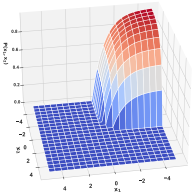
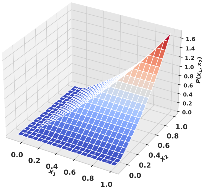
Setting

Pricing multi-asset options: compute $\mathbb{E}[P(\mathbf{X}_T)]$

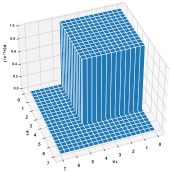
- $P(\cdot)$: payoff function (typically non-smooth), e.g., (K: the strike price)
  - Basket put $P(\mathbf{x}) = \max(\sum_{i=1}^{d} c_i e^{x_i} - K, 0)$, s.t. $c_i > 0, \sum_{i=1}^{d} c_i = 1$;
  - Rainbow (E.g., Call on min):
    $P(\mathbf{x}) = \max\left(\min\left(e^{x_1}, \ldots, e^{x_d}\right) - K, 0\right)$
  - Cash-or-nothing put: $P(\mathbf{x}) = \prod_{i=1}^{d} \mathbf{1}_{[0,K_i]}(e^{x_i})$.
- $\mathbf{X}_T$ is a d-dimensional ($d \geq 1$) vector of log-asset prices at time T, following a certain multivariate stochastic model with an affine structure (e.g., Lévy models).



(a) Basket put



(b) Call on min



(c) Cash-or-nothing put

Figure 1.1: Payoff functions illustration

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Challenges

1. Monte Carlo (MC) method (prevalent choice) has a rate of convergence independent of the problem's dimension and regularity of the payoff but can be very slow.
2. $P(\cdot)$ is non-smooth $\Rightarrow$ deteriorates convergence of deterministic quadrature.
3. The curse of dimensionality and other issues $\Rightarrow$ Most proposed Fourier pricing approaches efficient for only 1D and 2D options (Carr et al. 1999; Lewis 2001; Fang et al. 2009; Hurd et al. 2010; Ruijter et al. 2012),. . . .

Aim: Empower Fourier-based pricing methods of multi-asset options

1. C. Ben Hammouda et al. "Optimal Damping with Hierarchical Adaptive Quadrature for Efficient Fourier Pricing of Multi-Asset Options in Lévy Models". In: *Journal of Computational Finance* 27.3 (2024), pp. 43–86. (Michael's talk)
2. C. Ben Hammouda et al. "Quasi-Monte Carlo for Efficient Fourier Pricing of Multi-Asset Options". In: *arXiv preprint arXiv:2403.02832* (2024). (Today's talk)

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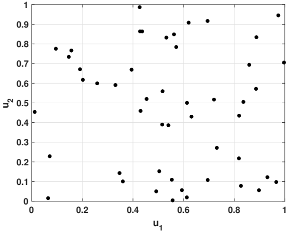
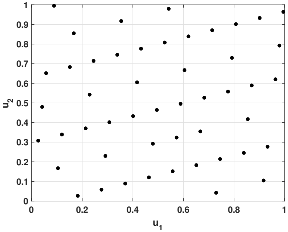
Numerical Integration Methods: Sampling in $[0,1]^2$

$$\mathrm{E}\left[P(\mathbf{X}(T))\right] = \int_{\mathbb{R}^d} P(\mathbf{x})\rho_{\mathbf{X}_T}(\mathbf{x})d\mathbf{x} \approx \sum_{m=1}^{M} \omega_m P\left(\mathbf{x}_m\right).$$

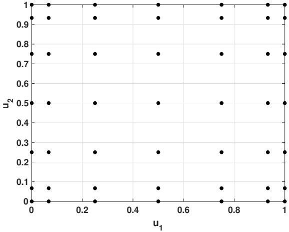
Monte Carlo (MC)



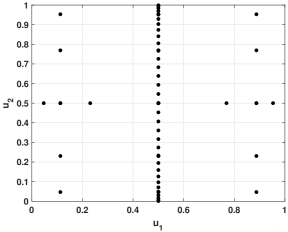
Quasi-Monte Carlo (QMC)



Tensor Product Quadrature



Adaptive Sparse Grids Quadrature



Fast Convergence: When Regularity Meets Structured Sampling

Monte Carlo (MC)

- (-) Slow convergence: $\mathcal{O}(M^{-\frac{1}{2}})$.
- (+) Rate independent of dimension and regularity of the integrand.

Quasi-Monte Carlo (QMC)

- Optimal Convergence: $\mathcal{O}(M^{-1})$ (Dick et al. 2013).
- Requires the **integrability of first mixed partial derivatives** of the integrand.
- Worst Case Convergence: $\mathcal{O}(M^{-1/2})$.

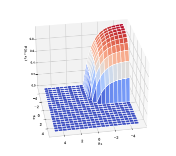
Tensor Product Quadrature

- Convergence: $\mathcal{O}(M^{-\frac{r}{d}})$ (Davis et al. 2007).
- $r > 0$ being the order of **bounded total derivatives** of the integrand.

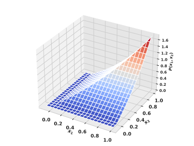
Adaptive Sparse Grids Quadrature

- Convergence: $\mathcal{O}(M^{-\frac{p}{2}})$ (Chen 2018).
- $p > 1$ is related to the order of **bounded weighted mixed (partial) derivatives** of the integrand.

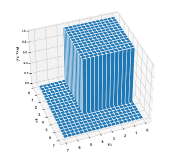
Challenge 1: Original problem is non smooth (low regularity)



(a) Basket Put



(b) Call on min

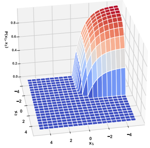


(c) Cash-or-nothing

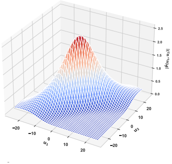
Solution: Uncover the available hidden regularity in the problem

1. Analytic smoothing (Bayer et al. 2018; Ben Hammouda et al. 2020): taking conditional expectations over subset of integration variables. ☹ Good choice not always trivial.
2. Numerical smoothing (Ben Hammouda et al. 2022):
   ☹ Additional computational work! Attractive when explicit smoothing or Fourier mapping not possible.
3. **Mapping the problem to the Fourier space (Today's talk)** (Ben Hammouda et al. 2024b; Ben Hammouda et al. 2024c).
   ⚠ Fourier transform of the density function (characteristic function) available/cheap to compute.

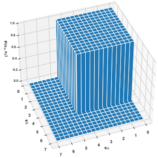
Better Regularity in the Fourier Space



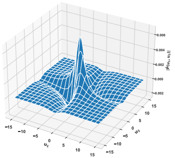
(a) Payoff: Basket put



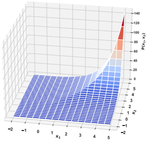
(b) Fourier Transform



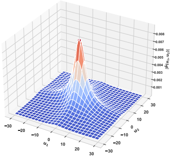
(a) Payoff:
Cash-or-nothing



(b) Fourier Transform



(a) Payoff: Call on min



(b) Fourier Transform

Fourier Pricing Formula in d Dimensions

Notation

- $\boldsymbol{\Theta}_m, \boldsymbol{\Theta}_p$: the model and payoff parameters, respectively;
- $\widehat{P}(\cdot)$: the Fourier transform of the payoff $P(\cdot)$;
- $\mathbf{X}_T$: vector of log-asset prices at time T, with extended characteristic function $\Phi_{\mathbf{X_T}}(\cdot)$;
- $\mathbf{R}$: vector of damping parameters ensuring integrability;
- δ_P: strip of regularity of $\widehat{P}(\cdot)$; δ_X: strip of regularity of $\Phi_{\mathbf{X_T}}(\cdot)$,

Assumption 1.1

1. $\mathbf{x} \mapsto P(\mathbf{x})$ *is continuous on* $\mathbb{R}^d$ *(Can be replaced by more regularity assumptions on the model).*
2. $\delta_P := \{\boldsymbol{R} \in \mathbb{R}^d : \boldsymbol{x} \mapsto e^{-\langle \boldsymbol{R}, \boldsymbol{x} \rangle} P(\boldsymbol{x}) \in L^1_{bc}(\mathbb{R}^d)$ *and* $\boldsymbol{y} \mapsto \widehat{P}(\boldsymbol{y} + \mathrm{i}\boldsymbol{R}) \in L^1(\mathbb{R}^d)\} \neq \varnothing.$
3. $\delta_X := \{\boldsymbol{R} \in \mathbb{R}^d : \boldsymbol{y} \mapsto | \Phi_{\boldsymbol{X}_T}(\boldsymbol{y} + \mathrm{i}\boldsymbol{R}) | < \infty, \forall\, \boldsymbol{y} \in \mathbb{R}^d\} \neq \varnothing.$

Proposition (Ben Hammouda et al. 2024b)

Under Assumptions 1, 2 and 3, and for $\mathbf{R} \in \delta_V := \delta_P \cap \delta_X$, *the value of the option price on* d *stocks is*

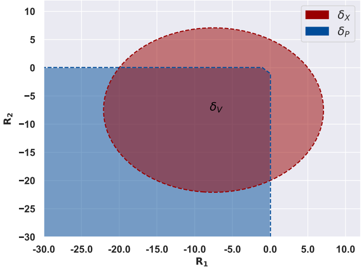
$$V(\boldsymbol{\Theta}_m, \boldsymbol{\Theta}_p) = e^{-rT}\mathbb{E}[P(\mathbf{X_T})] \tag{1}$$

$$= \int_{\mathbb{R}^d} \underbrace{(2\pi)^{-d} e^{-rT} \Re\left(\Phi_{\mathbf{X_T}}(\mathbf{y} + \mathrm{i}\mathbf{R}) \widehat{P}(\mathbf{y} + \mathrm{i}\mathbf{R})\right)}_{:= g(\mathbf{y}; \mathbf{R}, \boldsymbol{\Theta}_m, \boldsymbol{\Theta}_p)} d\mathbf{y}.$$

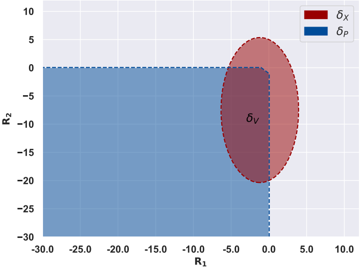
Challenge 2: The choice of the damping parameters

Damping parameters, $\mathbf{R}$, ensure integrability and control the regularity of the integrand.

Figure 1.6: Example of a **strip of analyticity** of the integrand of a 2D **call on min** option under **VG** model. Parameters: $\boldsymbol{\theta} = (-0.3, -0.3), \nu = 0.5, \boldsymbol{\Sigma} = \boldsymbol{I}_2$.



(a) $\boldsymbol{\sigma} = (0.2, 0.2)$



(b) $\boldsymbol{\sigma} = (0.2, 0.5)$

Challenge 2: The choice of the damping parameters

Damping parameters, $\mathbf{R}$, ensure integrability and control the regularity of the integrand.

Solution: (Ben Hammouda et al. 2024b) and Michael's talk

Based on contour integration error estimates: Parametric smoothing of the Fourier integrand via an (generic) optimization rule for the choice of damping parameters.

Near-Optimal Damping Rule (Ben Hammouda et al. 2024b)

We propose an optimization rule for choosing the damping parameters

$$\mathbf{R}^* := \mathbf{R}^*(\mathbf{\Theta}_m, \mathbf{\Theta}_p) = \arg\min_{\mathbf{R}\in\delta_V} \|g(\mathbf{u}; \mathbf{R}, \mathbf{\Theta}_m, \mathbf{\Theta}_p)\|_\infty$$
$$= \arg\min_{\mathbf{R}\in\delta_V} g(\mathbf{0}_{\mathbb{R}^d}; \mathbf{R}, \mathbf{\Theta}_m, \mathbf{\Theta}_p). \quad (2)$$

where $\mathbf{R}^* := (R_1^*, \dots, R_d^*)$ denotes the optimal damping parameters.

Challenge 3: Curse of dimensionality

1. Most of the existing Fourier approaches face hurdles in high-dimensional settings due to the tensor product (TP) structure of the commonly employed numerical quadrature techniques.
2. Complexity of (standard) TP quadrature to solve (1) ↗ exponentially with the number of underlying assets (Recall Convergence: $\mathcal{O}(M^{-\frac{r}{d}})$).

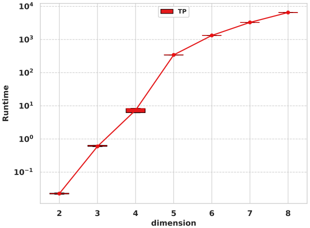


Figure 1.7: Call on min option under Normal Inverse Gaussian model: Runtime (in sec) versus dimension for TP for a relative error TOL = 10^{-2}.

Solution: Effective treatment of the high dimensionality

1. (Ben Hammouda et al. 2024b): Sparsification and dimension-adaptivity techniques to accelerate convergence **(Michael's talk)**.
2. (Ben Hammouda et al. 2024c): Quasi-Monte Carlo (QMC) with efficient domain transformation **(Today's talk)**.

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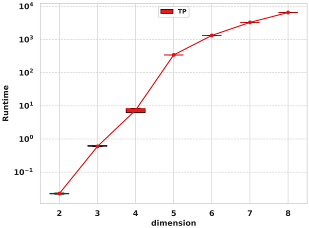

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Quasi-Monte Carlo (QMC): Need for Domain Transformation

Recall: our Fourier integrand is:

$$g(\mathbf{y};\mathbf{R}) = (2\pi)^{-d} e^{-rT} \Re\left(\Phi_{\mathbf{X_T}}(\mathbf{y}+\mathrm{i}\mathbf{R})\widehat{P}(\mathbf{y}+\mathrm{i}\mathbf{R})\right), \; \mathbf{y} \in \mathbb{R}^d, \; \mathbf{R} \in \delta_V := \delta_P \cap \delta_X$$

- Our Fourier integrand is in $\mathbb{R}^d$ **BUT** QMC constructions are restricted to the generation of low-discrepancy point sets on $[0,1]^d$.

  $\Rightarrow$ Need to transform the integration domain

- Using an inverse cumulative distribution function, we obtain the value of the option price on d stocks:

$$V(\mathbf{\Theta}_m, \mathbf{\Theta}_p) = \int_{\mathbb{R}^d} g(\mathbf{y}) \mathbf{dy} = \int_{[0,1]^d} \underbrace{\frac{g \circ \Psi^{-1}(\mathbf{u};\mathbf{\Lambda})}{\psi \circ \Psi^{-1}(\mathbf{u};\mathbf{\Lambda})}}_{=:\tilde{g}(\mathbf{u};\mathbf{\Lambda})} \mathbf{du}.$$

  - $\psi(\cdot;\mathbf{\Lambda})$: a probability density function (PDF) with parameters $\mathbf{\Lambda}$.
  - $\Psi(\cdot;\mathbf{\Lambda})$: the corresponding cumulative distribution function (CDF).

Randomized Quasi-Monte Carlo (RQMC)

- The transformed integration problem reads now:

$$V(\boldsymbol{\Theta}_m, \boldsymbol{\Theta}_p) = \int_{[0,1]^d} \underbrace{\frac{g \circ \Psi^{-1}(\mathbf{u}; \boldsymbol{\Lambda})}{\psi \circ \Psi^{-1}(\mathbf{u}; \boldsymbol{\Lambda})}}_{=: \tilde{g}(\mathbf{u}; \boldsymbol{\Lambda})} d\mathbf{u}. \tag{3}$$

- Once the choice of $\psi(\cdot; \boldsymbol{\Lambda})$ (respectively $\Psi^{-1}(\cdot; \boldsymbol{\Lambda})$) is determined, the RQMC estimator of (3) can be expressed as follows:

$$Q_{N,s}^{RQMC}[\tilde{g}] := \frac{1}{S} \sum_{i=1}^{S} \frac{1}{N} \sum_{n=1}^{N} \tilde{g}\left(u_n^{(s)}; \boldsymbol{\Lambda}\right), \tag{4}$$

  - $\{u_n\}_{n=1}^{N}$ is the sequence of deterministic QMC points
  - For $n = 1, \ldots, N$, $\{u_n^{(s)}\}_{s=1}^{S}$: obtained by an appropriate randomization of $\{u_n\}_{n=1}^{N}$, such that $\{u_n^{(s)}\}_{s=1}^{S} \overset{i.i.d}{\sim} U([0,1]^d)$.

- Why Randomization?
  - Practical error estimates based on the central limit theorem.

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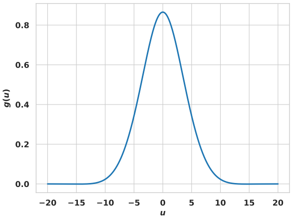
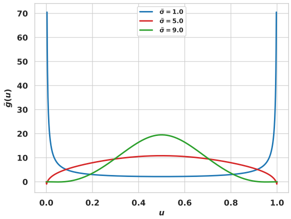
Challenge 4: Deterioration of QMC convergence if ψ or/and $\mathbf{\Lambda}$ are badly chosen

- **Observe:** The denominator of $\tilde{g}(\mathbf{u}) = \frac{g \circ \Psi^{-1}(\mathbf{u}; \mathbf{\Lambda})}{\psi \circ \Psi^{-1}(\mathbf{u}; \mathbf{\Lambda})}$ decays to 0 as $u_j \to 0, 1$ for $j = 1, \ldots, d$.
- The transformed integrand may have singularities near the boundary of $[0, 1]^d \Rightarrow$ Deterioration of QMC convergence.



(a) Original Fourier integrand (1) for call option under GBM



(b) Domain transformation for the integrand (1)

Questions

Q1: Which density to choose? Q2: How to choose its parameters?

How to choose $\psi(\cdot;\mathbf{\Lambda})$ (respectively $\Psi^{-1}(\cdot;\mathbf{\Lambda})$) and and its parameters, $\mathbf{\Lambda}$?

For $\boldsymbol{u} \in [0,1]^d, \boldsymbol{R} \in \delta_V$, the transformed Fourier integrand reads:

$$\tilde{g}(\boldsymbol{u}) = \frac{g \circ \Psi^{-1}(\mathbf{u};\mathbf{\Lambda})}{\psi \circ \Psi^{-1}(\mathbf{u};\mathbf{\Lambda})} = \frac{e^{-rT}}{(2\pi)^d} \Re\left[\widehat{P}(\Psi^{-1}(\boldsymbol{u}) + \mathrm{i}\boldsymbol{R}) \frac{\Phi_{\boldsymbol{X}_T}\left(\Psi^{-1}(\boldsymbol{u}) + \mathrm{i}\boldsymbol{R}\right)}{\psi\left(\Psi^{-1}(\boldsymbol{u})\right)}\right].$$

$\Rightarrow$ Sufficient to design the domain transformation to control the growth at the boundaries of the term $\frac{\Phi_{\boldsymbol{X}_T}(\Psi^{-1}(\boldsymbol{u})+\mathrm{i}\boldsymbol{R})}{\psi(\Psi^{-1}(\boldsymbol{u}))}$ (Conservative choice).

- The payoff Fourier transforms ($\widehat{P}(\cdot)$) decay at a polynomial rate.
- PDFs of the pricing models (light and semi-heavy tailed models), if they exist, are much smoother than the payoff $\Rightarrow$ the decay of their Fourier transforms (charactersitic functions) is faster the one of the payoff Fourier transform (Trefethen 1996; Cont et al. 2003).

How to choose $\psi(\cdot;\mathbf{\Lambda})$ (respectively $\Psi^{-1}(\cdot;\mathbf{\Lambda})$) and and its parameters, $\mathbf{\Lambda}$?

For $\boldsymbol{u} \in [0,1]^d, \boldsymbol{R} \in \delta_V$, the transformed Fourier integrand reads:

$$\begin{aligned}\tilde{g}(\boldsymbol{u}) &= \frac{g \circ \Psi^{-1}(\mathbf{u};\mathbf{\Lambda})}{\psi \circ \Psi^{-1}(\mathbf{u};\mathbf{\Lambda})} \\ &= \frac{e^{-rT}}{(2\pi)^d} \Re\left[\widehat{P}(\Psi^{-1}(\boldsymbol{u}) + \mathrm{i}\boldsymbol{R}) \frac{\Phi_{\boldsymbol{X}_T}\left(\Psi^{-1}(\boldsymbol{u}) + \mathrm{i}\boldsymbol{R}\right)}{\psi\left(\Psi^{-1}(\boldsymbol{u})\right)}\right].\end{aligned}$$

$\Rightarrow$ Sufficient to design the domain transformation to control the growth at the boundaries of the term $\frac{\Phi_{\boldsymbol{X}_T}(\Psi^{-1}(\boldsymbol{u})+\mathrm{i}\boldsymbol{R})}{\psi(\Psi^{-1}(\boldsymbol{u}))}$ (Conservative choice).

- The payoff Fourier transforms ($\widehat{P}(\cdot)$) decay at a polynomial rate.
- PDFs of the pricing models (light and semi-heavy tailed models), if they exist, are much smoother than the payoff $\Rightarrow$ the decay of their Fourier transforms (charactersitic functions) is faster the one of the payoff Fourier transform (Trefethen 1996; Cont et al. 2003).

Model-dependent Domain Transformation

Solution (Ben Hammouda et al. 2024c): Effective Domain Transformation

1. Choose the density $\psi(\cdot;\Lambda)$ to asymptotically follow the same functional form of the characteristic function.

Table 1: Extended characteristic function: $\Phi_{\mathbf{X}_T}(\mathbf{z}) = \exp(\mathrm{i}\mathbf{z}'\mathbf{X}_0)\exp(\mathrm{i}\mathbf{z}'\boldsymbol{\mu}T)\,\phi_{\mathbf{X}_T}(\mathbf{z})$, and choice of $\psi(\cdot)$.

$\phi_{\mathbf{X}_T}(\mathbf{z}), \mathbf{z} \in \mathbb{C}^d, \Im[\mathbf{z}] \in \delta_X$	$\psi(\mathbf{y};\boldsymbol{\Lambda}), \mathbf{y} \in \mathbb{R}^d$
GBM model: $\exp\left(-\frac{T}{2}\mathbf{z}'\boldsymbol{\Sigma}\mathbf{z}\right)$	**Gaussian** ($\boldsymbol{\Lambda} = \tilde{\boldsymbol{\Sigma}}$): $(2\pi)^{-\frac{d}{2}}(\det(\tilde{\boldsymbol{\Sigma}}))^{-\frac{1}{2}}\exp\left(-\frac{1}{2}(\mathbf{y}'\tilde{\boldsymbol{\Sigma}}^{-1}\mathbf{y})\right)$
VG model: $\left(1 - \mathrm{i}\nu\mathbf{z}'\boldsymbol{\theta} + \frac{1}{2}\nu\mathbf{z}'\boldsymbol{\Sigma}\mathbf{z}\right)^{-T/\nu}$	**Generalized Student's t** ($\boldsymbol{\Lambda} = (\tilde{\nu}, \tilde{\boldsymbol{\Sigma}})$): $\frac{\Gamma\left(\frac{\tilde{\nu}+d}{2}\right)(\det(\tilde{\boldsymbol{\Sigma}}))^{-\frac{1}{2}}}{\Gamma\left(\frac{\tilde{\nu}}{2}\right)\tilde{\nu}^{\frac{d}{2}}\pi^{\frac{d}{2}}}\left(1 + \frac{1}{\tilde{\nu}}\left(\mathbf{y}'\tilde{\boldsymbol{\Sigma}}\mathbf{y}\right)\right)^{-\frac{\tilde{\nu}+d}{2}}$
NIG model: $\exp\left(\delta T\left(\sqrt{\alpha^2 - \boldsymbol{\beta}'\boldsymbol{\Delta}\boldsymbol{\beta}} - \sqrt{\alpha^2 - (\boldsymbol{\beta}+\mathrm{i}\mathbf{z})'\boldsymbol{\Delta}(\boldsymbol{\beta}+\mathrm{i}\mathbf{z})}\right)\right)$	**Laplace** ($\boldsymbol{\Lambda} = \tilde{\boldsymbol{\Sigma}}$) and ($v = \frac{2-d}{2}$): $(2\pi)^{-\frac{d}{2}}(\det(\tilde{\boldsymbol{\Sigma}}))^{-\frac{1}{2}}\left(\frac{\mathbf{y}'\tilde{\boldsymbol{\Sigma}}^{-1}\mathbf{y}}{2}\right)^{\frac{v}{2}}K_v\left(\sqrt{2\mathbf{y}'\tilde{\boldsymbol{\Sigma}}^{-1}\mathbf{y}}\right)$

Notation:

- $\boldsymbol{\Sigma}$: Covariance matrix for the Geometric Brownian Motion (GBM) model.
- $\nu, \boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\Sigma}$: Variance Gamma (VG) model parameters.
- $\alpha, \boldsymbol{\beta}, \delta, \boldsymbol{\Delta}$: Normal Inverse Gaussian (NIG) model parameters.
- $\boldsymbol{\mu}$ is the martingale correction term.
- $K_v(\cdot)$: the modified Bessel function of the second kind.

Model-dependent Domain Transformation: Case of Independent Assets

Using independence: Observe $\frac{\phi_{\boldsymbol{X}_T}(\Psi^{-1}(\boldsymbol{u})+\mathrm{i}\boldsymbol{R})}{\psi(\Psi^{-1}(\boldsymbol{u}))} = \prod_{j=1}^{d} \frac{\phi_{X_T^j}(\Psi^{-1}(u_j)+\mathrm{i}R_j)}{\psi_j(\Psi^{-1}(u_j))}$

Solution (Ben Hammouda et al. 2024c): Effective Domain Transformation

1. Choose the density $\psi(\cdot;\boldsymbol{\Lambda})$ in the change of variable to asymptotically follow the same functional form of the extended characteristic function.
2. Select the parameters $\boldsymbol{\Lambda}$ to control the growth of the integrand near the boundary of $[0,1]^d$ i.e $\lim_{u_j \to 0,1} \tilde{g}(u_j) < \infty, j = 1, \dots, d$.

Table 2: Choice of $\psi(\mathbf{u};\boldsymbol{\Lambda}) \coloneqq \prod_{j=1}^{d} \psi_j(u_j;\boldsymbol{\Lambda})$ and conditions on $\boldsymbol{\Lambda}$ for GBM, (ii) VG and (iii) NIG. See (Ben Hammouda et al. 2024c) for the derivation.

Model	$\psi_j(y_j;\boldsymbol{\Lambda})$	Growth condition on $\boldsymbol{\Lambda}$
GBM	$\frac{1}{\sqrt{2\tilde{\sigma}_j{}^2}} \exp(-\frac{y_j^2}{2\tilde{\sigma}_j{}^2})$ (Gaussian)	$\tilde{\sigma}_j \geq \frac{1}{\sqrt{T}\sigma_j}$
VG	$\frac{\Gamma(\frac{\tilde{\nu}+1}{2})}{\sqrt{\tilde{\nu}\pi}\tilde{\sigma}_j \Gamma(\frac{\tilde{\nu}}{2})} \left(1 + \frac{y_j^2}{\tilde{\nu}\tilde{\sigma}_j{}^2}\right)^{-(\tilde{\nu}+1)/2}$ (t-Student)	$\tilde{\nu} \leq \frac{2T}{\nu} - 1$, $\tilde{\sigma}_j = \left(\frac{\nu\sigma_j^2\tilde{\nu}}{2}\right)^{\frac{T}{\nu-2T}} (\tilde{\nu})^{\frac{\nu}{4T-2\nu}}$
NIG	$\frac{\exp(-\frac{\lvert y_j \rvert}{\tilde{\sigma}_j})}{2\tilde{\sigma}_j}$ (Laplace)	$\tilde{\sigma}_j \geq \frac{1}{\delta T}$

⚠ In case of equality conditions, the integrand still decays at the speed of the payoff transform.

Should Correlation Be Considered in the Domain Transformation?

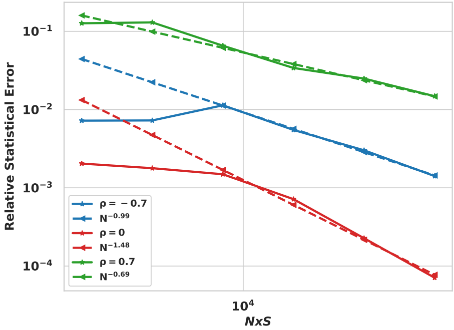


Figure 2.2: Two-dimensional call on the minimum option under the GBM model: Effect of the correlation parameter, ρ, on the convergence of RQMC.
For the domain transformation, we set $\tilde{\sigma}_j = \frac{1}{\sqrt{T}\sigma_j} = 5$, $j = 1, 2$. N: number of QMC points; $S = 32$: number of digital shifts.

Model-dependent Domain Transformation: Case of Correlated Assets

Challenge 5: Numerical Evaluation of the inverse CDF $\Psi^{-1}(\cdot)$

1. We can not evaluate the inverse CDF componentwise using the univariate inverse CDF as in the independent case $(\Psi_d^{-1}(\mathbf{u}) \neq (\Psi_1^{-1}(u_1), \ldots, \Psi_1^{-1}(u_d)))$.
2. The inverse CDF is not given in closed-form for most multivariate distributions, and its numerical approximation is generally computationally expensive.

Observe: For GBM model: If $\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I}_d) \Rightarrow \mathbf{X} = \tilde{\boldsymbol{L}}\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, \tilde{\boldsymbol{\Sigma}})$ ($\tilde{\boldsymbol{L}}$: Cholesky factor of $\tilde{\boldsymbol{\Sigma}}$) $\Rightarrow$ we have $\Psi_{nor,d}^{-1}(\mathbf{u}; \tilde{\boldsymbol{\Sigma}}) = \tilde{\boldsymbol{L}} \Psi_{nor,d}^{-1}(\mathbf{u}; \boldsymbol{I}_d) = \tilde{\boldsymbol{L}} \left(\Psi_{nor,1}^{-1}(u_1), \ldots, \Psi_{nor,1}^{-1}(u_d) \right)$

Solution: Avoid the expensive computation of the inverse CDF

1. We consider multivariate transformation densities, $\psi(\cdot, \boldsymbol{\Lambda})$, which belong to the class of normal mean-variance mixture distributions; i.e., for $\mathbf{X} \sim \psi(\cdot, \boldsymbol{\Lambda})$, we can write $\mathbf{X} = \boldsymbol{\mu} + W\mathbf{Z}$, with $\mathbf{Z} \sim \mathcal{N}_d(\mathbf{0}, \boldsymbol{\Sigma})$, and $W \geq 0$, independent of $\mathbf{Z}$.
2. We use the eigenvalue or Cholesky decomposition to eliminate the dependence structure.

Model-dependent Domain Transformation: Case of Correlated Assets

Challenge 5: Numerical Evaluation of the inverse CDF $\Psi^{-1}(\cdot)$

1. We can not evaluate the inverse CDF componentwise using the univariate inverse CDF as in the independent case $(\Psi_d^{-1}(\mathbf{u}) \neq (\Psi_1^{-1}(u_1), \ldots, \Psi_1^{-1}(u_d)))$.
2. The inverse CDF is not given in closed-form for most multivariate distributions, and its numerical approximation is generally computationally expensive.

Observe: For **GBM** model: If $\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I_d}) \Rightarrow \mathbf{X} = \tilde{\boldsymbol{L}}\mathbf{Z} \sim \mathcal{N}(\mathbf{0}, \tilde{\boldsymbol{\Sigma}})$ ($\tilde{\boldsymbol{L}}$: Cholesky factor of $\tilde{\boldsymbol{\Sigma}}$) $\Rightarrow$ we have $\Psi_{nor,d}^{-1}(\mathbf{u}; \tilde{\boldsymbol{\Sigma}}) = \tilde{\boldsymbol{L}}\Psi_{nor,d}^{-1}(\mathbf{u}; \boldsymbol{I_d}) = \tilde{\boldsymbol{L}}\left(\Psi_{nor,1}^{-1}(u_1), \ldots, \Psi_{nor,1}^{-1}(u_d)\right)$

Solution: Avoid the expensive computation of the inverse CDF

1. We consider multivariate transformation densities, $\psi(\cdot, \boldsymbol{\Lambda})$, which belong to the class of normal mean-variance mixture distributions; i.e., for $\mathbf{X} \sim \psi(\cdot, \boldsymbol{\Lambda})$, we can write $\mathbf{X} = \boldsymbol{\mu} + W\mathbf{Z}$, with $\mathbf{Z} \sim \mathcal{N}_d(\mathbf{0}, \boldsymbol{\Sigma})$, and $W \geq 0$, independent of $\mathbf{Z}$.
2. We use the eigenvalue or Cholesky decomposition to eliminate the dependence structure.

Illustration

- **GBM** model : Using $\tilde{\boldsymbol{L}}\Psi^{-1}_{nor,d}(\mathbf{u};\boldsymbol{I_d}) = \tilde{\boldsymbol{L}}\left(\Psi^{-1}_{nor,1}(u_1),\ldots,\Psi^{-1}_{nor,1}(u_d)\right)$ ($\tilde{\boldsymbol{L}}$: Cholesky factor of $\tilde{\boldsymbol{\Sigma}}$), we obtain

$$\int_{\mathbb{R}^d} g(\boldsymbol{y})d\boldsymbol{y} = \int_{[0,1]^d} \frac{g\left(\tilde{\boldsymbol{L}}\Psi^{-1}_{nor,d}(\boldsymbol{u};\boldsymbol{I}_d)\right)}{\psi^{nor}\left(\tilde{\boldsymbol{L}}\Psi^{-1}_{nor,d}(\boldsymbol{u};\boldsymbol{I}_d)\right)}d\boldsymbol{u},$$

- **VG** model: **Observe:** If $\mathbf{Z} \sim \mathcal{N}(\mathbf{0},\tilde{\boldsymbol{\Sigma}}), Y \sim \chi^2(\tilde{\nu}) \Rightarrow$ $\mathbf{X} = \mathbf{Z} \times \frac{\sqrt{\tilde{\nu}}}{\sqrt{Y}} \sim t_d(\tilde{\nu},\mathbf{0},\tilde{\boldsymbol{\Sigma}})$, with $\mathbf{Z}, Y$ independent
$\Rightarrow$ we obtain (see Proposition 3.4 in (Ben Hammouda et al. 2024c))

$$\int_{\mathbb{R}^d} g(\mathbf{u})\mathbf{du} = \int_0^{+\infty}\left(\int_{[0,1]^d} \frac{g\left(\frac{\tilde{\boldsymbol{L}}\cdot\Psi^{-1}_{nor,d}(\mathbf{u};\boldsymbol{I_d})}{\sqrt{y}}\right)}{\psi_{\text{stu}}\left(\frac{\tilde{\boldsymbol{L}}\cdot\Psi^{-1}_{nor,d}(\mathbf{u};\boldsymbol{I_d})}{\sqrt{y}}\right)}\mathbf{du}\right)\rho_Y(y)dy$$

  - $t_d(\tilde{\nu},\mathbf{0},\tilde{\boldsymbol{\Sigma}})$: generalized t-student distribution.
  - $\rho_Y(\cdot)$: density of $\chi^2(\tilde{\nu})$ (chi-squared) distribution.
  - $\tilde{\boldsymbol{L}}$: Cholesky factor of $\tilde{\nu} \times \tilde{\boldsymbol{\Sigma}}$

Model-dependent Domain Transformation: Case of Correlated Assets

Solution (Ben Hammouda et al. 2024c): Effective Domain Transformation

1. Choose the density $\psi(\cdot;\boldsymbol{\Lambda})$ in the change of variable to asymptotically follow the same functional form of the extended characteristic function.
2. Select the parameters $\boldsymbol{\Lambda}$ to control the growth of the integrand near the boundary of $[0,1]^d$ i.e $\lim_{u_j \to 0,1} \tilde{g}(u_j) < \infty, j = 1, \dots, d$.

Table 3: Choice of $\psi(\mathbf{u};\boldsymbol{\Lambda}) := \prod_{j=1}^{d} \psi_j(u_j;\boldsymbol{\Lambda})$ and conditions on $\boldsymbol{\Lambda}$ for GBM, (ii) VG and (iii) NIG. See (Ben Hammouda et al. 2024c) for the derivation.

Model	$\psi(\boldsymbol{y};\boldsymbol{\Lambda})$	Growth condition on $\boldsymbol{\Lambda}$
GBM	Gaussian: $(2\pi)^{-\frac{d}{2}}(\det(\tilde{\boldsymbol{\Sigma}}))^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\boldsymbol{y}'\tilde{\boldsymbol{\Sigma}}^{-1}\boldsymbol{y})\right)$	$T\boldsymbol{\Sigma} - \tilde{\boldsymbol{\Sigma}}^{-1} \geq 0$
VG	Generalized Student's t: $\frac{\Gamma\left(\frac{\tilde{\nu}+d}{2}\right)(\det(\tilde{\boldsymbol{\Sigma}}))^{-\frac{1}{2}}}{\Gamma\left(\frac{\tilde{\nu}}{2}\right)\tilde{\nu}^{\frac{d}{2}}\pi^{\frac{d}{2}}}\left(1 + \frac{1}{\tilde{\nu}}\left(\boldsymbol{y}'\tilde{\boldsymbol{\Sigma}}\boldsymbol{y}\right)\right)^{-\frac{\tilde{\nu}+d}{2}}$	$\tilde{\nu} = \frac{2T}{\nu} - d$, and $\boldsymbol{\Sigma} - \tilde{\boldsymbol{\Sigma}}^{-1} \geq 0$ **or** $\tilde{\nu} \leq \frac{2T}{\nu} - d$, and $\tilde{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma}^{-1}$
NIG	Laplace $(v = \frac{2-d}{2})$: $(2\pi)^{-\frac{d}{2}}(\det(\tilde{\boldsymbol{\Sigma}}))^{-\frac{1}{2}}\left(\frac{\boldsymbol{y}'\tilde{\boldsymbol{\Sigma}}^{-1}\boldsymbol{y}}{2}\right)^{\frac{v}{2}} K_v\left(\sqrt{2\boldsymbol{y}'\tilde{\boldsymbol{\Sigma}}^{-1}\boldsymbol{y}}\right)$	$\delta^2 T^2 \boldsymbol{\Delta} - 2\tilde{\boldsymbol{\Sigma}}^{-1} \geq 0$

Illustration: Case of Correlated Assets

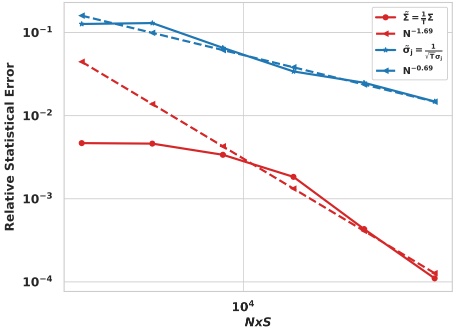
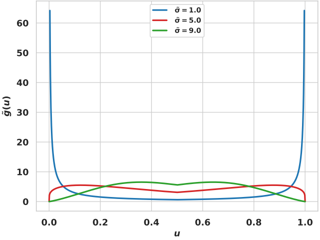
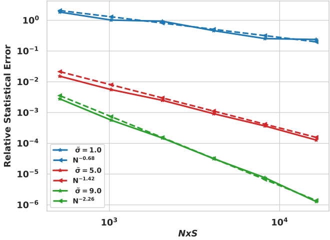


Figure 2.3: Two-dimensional call on the minimum option under the GBM model: Effect of the correlation parameter, ρ, on the convergence of RQMC. N: number of QMC points; $S = 32$: number of digital shifts.

Effect of Domain Transformation on RQMC Convergence



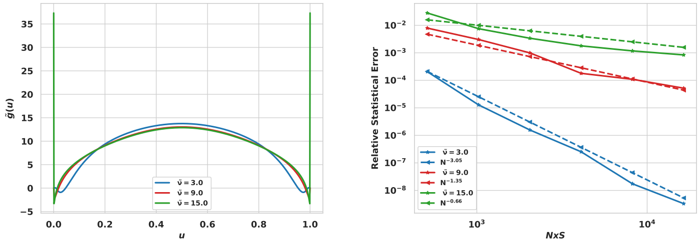
(a)



(b)

Figure 3.1: Call option under the NIG model: Effect of the parameter $\tilde{\sigma}$ of the Laplace PDF on
(a) the shape of the transformed integrand $\tilde{g}(u)$ and
(b) convergence of the relative statistical error of RQMC
N: number of QMC points; $S = 32$: number of digital shifts.
Boundary growth condition: $\tilde{\sigma} \geq \frac{1}{T\delta} = 5$.

Effect of Domain Transformation on RQMC Convergence



(a) (b)

Figure 3.2: Call option under the VG model: Effect of the parameter $\tilde{\nu}$ of the t-student PDF on

(a) the shape of the transformed integrand $\tilde{g}(u)$ and

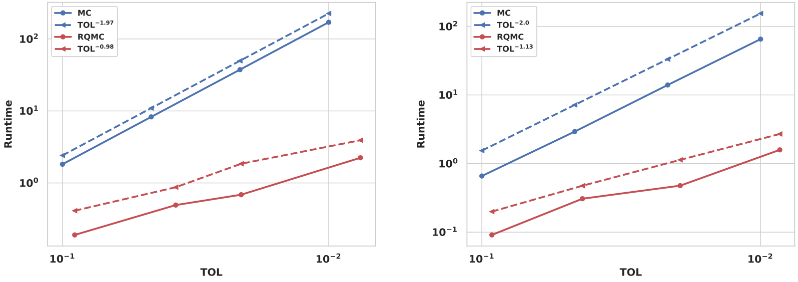
(b) convergence of the RQMC error

N: number of QMC points; $S = 32$: number of digital shifts.

Boundary growth condition: $\tilde{\nu} \leq \frac{2T}{\nu} - 1 = 9$

RQMC In Fourier Space vs MC in Physical Space

Figure 3.3: Average runtime in seconds with respect to relative tolerance levels TOL: Comparison of RQMC in the Fourier space (with optimal damping parameters and appropriate domain transformation) and MC in the physical space.

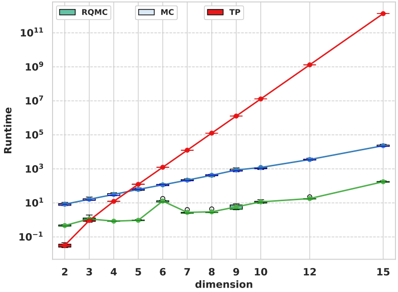


(a) 6D-VG call on min

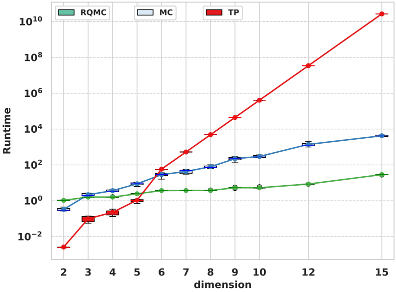
(b) 6D-NIG call on min

Comparison of the Different Methods

Figure 3.4: Call on min option: Runtime (in sec) versus dimensions to reach a relative error, $TOL = 10^{-2}$. RQMC in the Fourier space (with optimal damping parameters and appropriate domain transformation), TP in the Fourier space with optimal damping parameters, and MC in the physical space.



(a) NIG model with:
$\alpha = 15, \beta_j = -3, \delta = 0.2, \boldsymbol{\Delta} = \boldsymbol{I}_d,$
$\tilde{\sigma}_j = \sqrt{\frac{2}{\delta^2 T^2}}$



(b) VG model with:
$\sigma_j = 0.2, \theta_j = -0.3, \nu = 0.2, \boldsymbol{\Sigma} = \boldsymbol{I}_d,$
$\tilde{\nu} = \frac{2T}{\nu} - d, \tilde{\sigma}_j = \frac{1}{\sigma_j}.$

Conclusion

1. We empower Fourier pricing methods of multi-asset options by employing QMC with an appropriate domain transformation.
2. We desing a practical (model dependent) domain transformation strategy that prevents singularities near boundaries, ensuring the integrand retains its regularity for faster QMC convergence in the Fourier space.
3. The designed QMC-based Fourier pricing approach outperforms the MC (in physical domain) and tensor product quadrature (in Fourier space) for pricing multi-asset options across up to 15 dimensions.
4. Accompanying code for the paper can be found here: Git repository: Quasi-Monte-Carlo-for-Efficient-Fourier-Pricing-of-Multi-Asset-Options

Related References

Thank you for your attention!

1. **C. Ben Hammouda et al. "Quasi-Monte Carlo for Efficient Fourier Pricing of Multi-Asset Options". In: *arXiv preprint arXiv:2403.02832* (2024)**
2. C. Ben Hammouda et al. "Optimal Damping with Hierarchical Adaptive Quadrature for Efficient Fourier Pricing of Multi-Asset Options in Lévy Models". In: *Journal of Computational Finance* 27.3 (2024), pp. 43–86
3. C. Ben Hammouda et al. "Numerical smoothing with hierarchical adaptive sparse grids and quasi-Monte Carlo methods for efficient option pricing". In: *Quantitative Finance* (2022), pp. 1–19
4. C. Ben Hammouda et al. "Hierarchical adaptive sparse grids and quasi-Monte Carlo for option pricing under the rough Bergomi model". In: *Quantitative Finance* 20.9 (2020), pp. 1457–1473

References I

[1] Christian Bayer, Markus Siebenmorgen, and Rául Tempone. "Smoothing the payoff for efficient computation of basket option pricing.". In: *Quantitative Finance* 18.3 (2018), pp. 491–505.

[2] C. Ben Hammouda, C. Bayer, and R. Tempone. "Hierarchical adaptive sparse grids and quasi-Monte Carlo for option pricing under the rough Bergomi model". In: *Quantitative Finance* 20.9 (2020), pp. 1457–1473.

[3] C. Ben Hammouda, C. Bayer, and R. Tempone. "Multilevel Monte Carlo combined with numerical smoothing for robust and efficient option pricing and density estimation". In: *arXiv preprint arXiv:2003.05708, to appear in SIAM Journal on Scientific Computing* (2024).

References II

[4] C. Ben Hammouda, C. Bayer, and R. Tempone. "Numerical smoothing with hierarchical adaptive sparse grids and quasi-Monte Carlo methods for efficient option pricing". In: *Quantitative Finance* (2022), pp. 1–19.

[5] C. Ben Hammouda et al. "Optimal Damping with Hierarchical Adaptive Quadrature for Efficient Fourier Pricing of Multi-Asset Options in Lévy Models". In: *Journal of Computational Finance* 27.3 (2024), pp. 43–86.

[6] C. Ben Hammouda et al. "Quasi-Monte Carlo for Efficient Fourier Pricing of Multi-Asset Options". In: *arXiv preprint arXiv:2403.02832* (2024).

[7] Peter Carr and Dilip Madan. "Option valuation using the fast Fourier transform". In: *Journal of computational finance* 2.4 (1999), pp. 61–73.

References III

[8] Peng Chen. "Sparse quadrature for high-dimensional integration with Gaussian measure". In: *ESAIM: Mathematical Modelling and Numerical Analysis* 52.2 (2018), pp. 631–657.

[9] Rama Cont and Peter Tankov. *Financial Modelling with Jump Processes*. Chapman and Hall/CRC, 2003.

[10] Philip J Davis and Philip Rabinowitz. *Methods of numerical integration*. Courier Corporation, 2007.

[11] Josef Dick, Frances Y Kuo, and Ian H Sloan. "High-dimensional integration: the quasi-Monte Carlo way". In: *Acta Numerica* 22 (2013), pp. 133–288.

[12] Ernst Eberlein, Kathrin Glau, and Antonis Papapantoleon. "Analysis of Fourier transform valuation formulas and applications". In: *Applied Mathematical Finance* 17.3 (2010), pp. 211–240.

References IV

[13] Fang Fang and Cornelis W Oosterlee. "A novel pricing method for European options based on Fourier-cosine series expansions". In: *SIAM Journal on Scientific Computing* 31.2 (2009), pp. 826–848.

[14] Thomas R Hurd and Zhuowei Zhou. "A Fourier transform method for spread option pricing". In: *SIAM Journal on Financial Mathematics* 1.1 (2010), pp. 142–157.

[15] Alan L Lewis. "A simple option formula for general jump-diffusion and other exponential Lévy processes". In: *Available at SSRN 282110* (2001).

[16] M. J. Ruijter and Cornelis W Oosterlee. "Two-dimensional Fourier cosine series expansion method for pricing financial options". In: *SIAM Journal on Scientific Computing* 34.5 (2012), B642–B671.

References V

[17] Lloyd Nicholas Trefethen. “Finite difference and spectral methods for ordinary and partial differential equations”. In: (1996).