Semi-static variance-optimal hedging with self-exciting jumps

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Motivation

1. Already existing literature on semi-static variance-optimal hedging, see Di Tella *et al.* 2019, Di Tella *et al.* 2020, but gap in research concerning models with self-exciting jumps.
2. Introduction of new financial derivatives, e.g. variance swaps $\longrightarrow$ necessity of studying hedging strategies in richer and more realistic models.
3. Interest in risk-management of variance swaps in the context of energy markets: speculation on volatility fluctuations on commodities markets (Gold, Oil, Natural Gas).

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The model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where $\mathbb{P}$ is the risk-neutral measure.

Log-price dynamics

Let X be the log-price of a stock, $S = e^X$,

$$\mathrm{d}X_t = \Big(-\frac{1}{2}V_t - (e^{\gamma+\delta^2/2} - 1)\lambda_t\Big)\mathrm{d}t + \sqrt{V_t}\,\mathrm{d}W_t^{(1)} + \mathrm{d}\Big(\sum_{i=1}^{N_t}\eta_i^X\Big), \quad X_0 = x_0 \in \mathbb{R}$$

$$\mathrm{d}V_t = \beta_v(\alpha_v - V_t)\mathrm{d}t + \sigma_v\sqrt{V_t}\,\mathrm{d}W_t^{(2)}, \qquad V_0 = v_0 > 0$$

$$\mathrm{d}\lambda_t = \beta_\lambda(\alpha_\lambda - \lambda_t)\,\mathrm{d}t + \mathrm{d}\Big(\sum_{i=1}^{N_t}\eta_i^\lambda\Big), \qquad \lambda_0 = \lambda_0 > 0.$$

- The log-price is given by a jump-diffusion dynamics.
- The volatility V is a CIR process (Heston-like volatility)

$$\mathrm{d}\langle W^{(1)}, W^{(2)}\rangle_t = \rho \in [-1, 1], \qquad \alpha_v, \beta_v, \sigma_v > 0, 2\alpha_v\beta_v \geq \sigma_v^2.$$

- λ is the intensity of a Hawkes process N: self-excitation

$$\alpha_\lambda, \beta_\lambda > 0, \qquad \eta_i^X \sim \mathcal{N}(\gamma, \delta^2), i.i.d, \qquad \eta_i^\lambda \sim \mathrm{Exp}(\zeta), i.i.d.$$

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Motivations

- Empirical studies have shown the presence of “jump clustering” in different markets: equities, see Aït-Sahalia *et al.* 2015, commodities, see Filimonov *et al.* 2014, energy, see Herrera and González 2014.
- Especially in commodity markets, “At least 60-70 % of commodity price changes are now due to self-generated activities rather than novel information”, Filimonov *et al.* 2014.
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The model

Properties

- The model is a generalization of the Heston model: when $\lambda \equiv 0$, our model coincides with Heston's.
- Under appropriate conditions on the parameters, the price $S = e^X$ is a square-integrable martingale.
- The model is affine, i.e., it is a Markov process and its Laplace function is exponentially-affine in the current state:

$$\mathbb{E}[e^{u^\top (X_T, V_T, \lambda_T)} | \mathcal{F}_t] = \exp(\Phi(u, T-t) + \Psi(u, T-t)^\top (X_t, V_t, \lambda_t)),$$

  where $(\mathcal{F}_t)_{t \in [0,T]}$ is the right continuous filtration such that (X, V, λ) is adapted →**closed formula for the Laplace transform.**

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The hedging problem

Target and contingent claims

- **Target claim**: Variance swap, financial derivative used to hedge or speculate on the magnitude of a price movement of an underlying asset.

$$\eta_T^0 = [X, X]_T - k,$$

  where k is the so-called swap rate, i.e., $k = \mathbb{E}[[X, X]_T]$ so that $\mathbb{E}[\eta_T^0] = 0$ and the contract is zero at inception.
- The set of **contingent claims** used to hedge is a basket $\boldsymbol{\eta} = (\eta^1, \dots, \eta^d)$ of European options written on S.

Variance swaps in energy markets

- Variance swaps contracts are commonly traded in equity markets (*S&P* index), **but** a remarkable interest has grown in recent years also in commodity markets.
- The recent developments occurred in both Oil and Natural Gas prices evolution stressed the need of accurate and reliable hedging strategies against market wild volatility fluctuations.
- Literature on variance swaps in energy markets: Prokopczuk *et al.* 2017, Swishchuk 2013, Carr and Corso 2001, Trolle and Schwartz 2010.

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Hedging problem
General setting

- $(\Omega, \mathcal{F}, \mathbb{P})$ probability space, where $\mathbb{P}$ is the risk-neutral measure.
- $\mathbb{E}[\cdot]$ denotes the expectation under the risk neutral probability $\mathbb{P}$.
- $[0, T]$ fixed time interval, $T > 0$.
- S models the price process of a tradable asset.
- $S \in L^2(\mathbb{P})$ (square-integrable) and it is a martingale.

Ingredients for hedging

- A **target claim** $\eta^0 \in L^2(\mathbb{P})$, which we want to hedge (e.g. variance swap).
- Fixed **basket of contingent claims** $\boldsymbol{\eta} := (\eta^1, \dots, \eta^d)$, $\eta^j \in L^2(\mathbb{P})$ for $j = 1, \dots, d$ (e.g. European options).

The hedging problem

Semi-static variance-optimal hedging problem

Semi-static variance-optimal hedging

A *semi-static variance-optimal* hedge is $(\vartheta, \boldsymbol{\nu}) \in L^2(S) \times \mathbb{R}^d$ with initial capital c of claim η^0 is solution of the minimization problem

$$\varepsilon^2 = \min_{\boldsymbol{\nu} \in \mathbb{R}^d, \vartheta \in L^2(S), c \in \mathbb{R}} \mathbb{E}\left[\left(c - \overbrace{\mathbb{E}[\boldsymbol{\nu}^\top \boldsymbol{\eta}]}^{\text{cost of the static part}} + \underbrace{\int_0^T \vartheta_s \, dS_s}_{\text{dynamic position}} - \eta^0 - \overbrace{\boldsymbol{\nu}^\top \boldsymbol{\eta}}^{\text{static position}} \right)^2\right],$$

where $L^2(S) := \left\{\vartheta \text{ predictable: } \mathbb{E}\left[\int_0^T |\vartheta_t|^2 \, d\langle S, S\rangle_t\right] < \infty\right\}.$

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A *semi-static* *variance-optimal* hedge is $(\vartheta, \boldsymbol{\nu}) \in L^2(S) \times \mathbb{R}^d$ with initial capital c of claim η^0 is solution of the minimization problem

$$\begin{cases} \varepsilon^2(\boldsymbol{\nu}) = \min_{\vartheta \in L^2(S), c \in \mathbb{R}} \mathbb{E}\left[\left(c - \mathbb{E}[\boldsymbol{\nu}^\top \boldsymbol{\eta}] + \int_0^T \vartheta_s \, \mathrm{d}S_s - \overbrace{(\eta^0 - \boldsymbol{\nu}^\top \boldsymbol{\eta})}^{\text{inner pb target claim}} \right)^2\right] & \text{(inner)} \\ \varepsilon^2 = \min_{\boldsymbol{\nu} \in \mathbb{R}^d} \varepsilon^2(\boldsymbol{\nu}) & \text{(outer)} \end{cases}$$

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- Inner problem is a classic variance-optimal hedging problem, Föllmer and Sondermann 1986 they solve it in the dynamic case.
- Outer problem is a finite-dimensional quadratic optimization problem.

How to solve the problem?

Galtchouk-Kunita–Watanabe decomposition

Fully dynamic variance-optimal hedging

Consider a fully dynamic variance-optimal hedging problem for a claim Y and a stock price S:

$$\varepsilon^2 = \min_{\vartheta \in L^2(S), c \in \mathbb{R}} \mathbb{E}\left[\left(c + \int_0^T \vartheta_s \, dS_s - Y_T\right)^2\right].$$

If both Y and S are **square-integrable martingales**, the variance-optimal hedging problem is solved by the Galtchouk-Kunita–Watanabe decomposition (ϑ, L)

$$Y_\cdot = \overbrace{c}^{\text{initial capital}} + \underbrace{\int_0^\cdot \underbrace{\vartheta_t}_{\text{investment strategy}} \mathrm{d}S_t}_{\text{hedgeable risk}} + \overbrace{L_\cdot}^{\text{residual unheadgeable risk}}.$$

where L is a local martingale orthogonal to S.

How to solve the problem?

General hedging strategies

Theorem 2.3, Di Tella *et al.* 2020

Let (ϑ^j, L^j) are the GKW decomposition of η^j with respect to S for $j = 0, \dots, d$, then the hedging problem is

$$\begin{cases} \varepsilon^2(\boldsymbol{\nu}) = \min_{\vartheta \in L^2(S), c \in \mathbb{R}} \mathbb{E}\left[\left(c - \mathbb{E}[\boldsymbol{\nu}^\top \boldsymbol{\eta}] + \int_0^T \vartheta_s \, \mathrm{d}S_s - (\eta^0 - \boldsymbol{\nu}^\top \boldsymbol{\eta})\right)^2\right] & \text{(inner)} \\ \varepsilon^2 = \min_{\boldsymbol{\nu} \in \mathbb{R}^d} A - 2\boldsymbol{\nu}^\top B + \boldsymbol{\nu}^\top C \boldsymbol{\nu} & \text{(outer)} \end{cases}$$

where $A := \mathbb{E}[\langle L^0, L^0 \rangle_T], B^j := \mathbb{E}[\langle L^0, L^j \rangle_T], C^{ij} := \mathbb{E}[\langle L^i, L^j \rangle_T], i, j = 1, \dots, d.$

Under a **non-redundancy condition**, C is invertible and the unique hedging strategy is given by

$$\boldsymbol{\nu}^* = C^{-1} B, \quad \vartheta^* = \vartheta^0 - \sum_{j=1}^d \nu_j^* \vartheta^j, \quad c^* = \mathbb{E}[\eta^0]$$

and the heging error by

$$\varepsilon^2(\boldsymbol{\nu}^*) = A - B^T C^{-1} B.$$

How to compute ϑ, A, B, C?

Practical steps

1. Combine GKW decomposition and Fourier representation of the claims.
   - Consider a claim Y_T having payoff $h(X_T)$, then its payoff can be written as

$$h(x) = \int_{\mathcal{S}(R)} \exp(u^\top x)\zeta(\mathrm{d}u)$$

for $\mathcal{S}(R) = \{R + iy, y \in \mathbb{R}\}, \zeta(\mathrm{d}u) := \frac{1}{2\pi i}\hat{h}(u)\mathrm{d}u$, where $\hat{h}(u)$ is the Laplace-transform of h.

European Calls. $h(x) = (e^x - K)^+, \quad \widehat{h}(u) = \frac{1}{2\pi i}\frac{K^{1-u}}{u(u-1)}, \quad R > 1.$

   - Instead of the GKW decomposition of η^j, consider $(\vartheta(u), L(u)), u \in \mathbb{C}$, the GKW decomposition of $\mathbb{E}[e^{uX}|\mathcal{F}_t] \in L^2(\mathbb{P})$. Following Di Tella *et al.* 2019, one can write for $i, j = 1, \ldots, d$

$$\vartheta^j = \int_{\mathcal{S}(R_j)} \vartheta(u)\,\zeta^j(\mathrm{d}u), \quad B^j = \int_0^T \int_{\mathcal{S}(R_j)} \mathbb{E}[\langle L^0, L(u_j)\rangle_t]\zeta^j(\mathrm{d}u_j)\mathrm{d}t,$$

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2. Exploit the results in Di Tella *et al.* 2019, Corollary 3.1, where they obtain semi-explicit expressions for

$$\vartheta^0, \quad \vartheta^j, \quad \langle L^0, L^0\rangle_t, \quad \langle L^0, L(u)\rangle_t, \quad \langle L(u), L(u)\rangle_t,$$

in the case of affine models.

Hedging strategy

Computations in our setting

1. Instead of the contingent claims η^j consider the Laplace-transform $\mathbb{E}[e^{uX_T}|\mathcal{F}_t]$, for $u \in \mathbb{C}$.

Proposition 1

Given $\mathbf{u} = (u_1, u_2, u_3) \in \mathcal{D} \subseteq \mathbb{C}^3$, *the conditional Laplace transform of* (X_T, V_T, λ_T) *given the* $\mathcal{F}_t$, *is given by:*

$$\mathbb{E}[\exp(u_1 X_T + u_2 V_T + u_3 \lambda_T)\,|\mathcal{F}_t] = \exp(\phi(T-t,\mathbf{u}) + u_1 X_t + \psi(T-t,\mathbf{u}) V_t + \chi(T-t,\mathbf{u})\lambda_t)$$

where

$$\begin{cases} \frac{\partial \phi}{\partial t}(t,\mathbf{u}) = \alpha_V \beta_V \psi(t,\mathbf{u}) + \alpha_\lambda \beta_\lambda \chi(t,\mathbf{u}), & \phi(0,\mathbf{u}) = 0, \\ \frac{\partial \psi}{\partial t}(t,\mathbf{u}) = -\frac{1}{2}u_1 + \frac{1}{2}u_1^2 - \beta_V \psi(t,\mathbf{u}) + \rho \sigma_V u_1 \psi(t,\mathbf{u}) + \frac{1}{2}\sigma_V^2 \psi(t,\mathbf{u})^2, & \psi(0,\mathbf{u}) = u_2, \\ \frac{\partial \chi}{\partial t}(t,\mathbf{u}) = -\beta_\lambda \chi(t,\mathbf{u}) - (e^{\gamma+\delta^2/2} - 1)u_1 + e^{\gamma u_1 + \delta^2 u_1^2/2}\frac{\zeta}{\zeta - \chi(t,\mathbf{u})} - 1, & \chi(0,\mathbf{u}) = u_3. \end{cases}$$

2. Exploit the results in Di Tella *et al.* 2019, Corollary 3.1 and the combination of GKW decomposition and Fourier representation.

ϑ

$$\vartheta_t^0 = \frac{\lambda_t}{S_{t-}(V_t + \bar{\kappa}\lambda_t)} F(t)$$
$$\vartheta_t^j = \int_{\mathcal{S}(R_j)} \frac{1}{S_{t-}(V_t + \bar{\kappa}\lambda_t)} \Big(f_{t-}(u_j) G_1(t) + f_{t-}(u_j)\lambda_t G_2(t) \Big) \zeta^j(\mathrm{d}u_j),$$

where $\bar{\kappa}$ constant and $f_{t-}(u_j) = e^{\phi_{T-t}(u_j,0,0)+u_j X_{t-}+\psi_{T-t}(u_j,0,0)V_{t-}+\chi_{T-t}(u_j,0,0)\lambda_{t-}}$ with F, G_1, G_2 deterministic functions of t, u_j and of the model's parameters

A, B, C

$$A = \int_0^T A_1(t)\mathbb{E}\left[\frac{V_t^2}{V_t + \lambda_t \bar{\kappa}}\right] + A_2(t)\mathbb{E}\left[\frac{\lambda_t V_t}{V_t + \lambda_t \bar{\kappa}}\right] + A_3(t)\mathbb{E}\left[\frac{\lambda_t^2}{V_t + \lambda_t \bar{\kappa}}\right] \mathrm{d}t,$$

$$B^j = \int_0^T \int_{\mathcal{S}(R_j)} B_1(t, u_j)\mathbb{E}\left[\frac{f_{t-}(u_j)\lambda_t V_t}{V_t + \lambda_t \bar{\kappa}}\right] + B_2(t, u_j)\mathbb{E}\left[\frac{f_{t-}(u_j)\lambda_t^2}{V_t + \lambda_t \bar{\kappa}}\right] \zeta^j(\mathrm{d}u_j)\mathrm{d}t,$$

$$\begin{aligned} C^{i,j} = \int_0^T \int_{\mathcal{S}(R_i)} \int_{\mathcal{S}(R_j)} & C_1(t, u_j, u_i)\mathbb{E}\left[\frac{f_{t-}(u_i) f_{t-}(u_j) V_t^2}{V_t + \lambda_t \bar{\kappa}}\right] \\ & + C_2(t, u_j, u_i)\mathbb{E}\left[\frac{f_{t-}(u_i) f_{t-}(u_j)\lambda_t V_t}{V_t + \lambda_t \bar{\kappa}}\right] \\ & + C_3(t, u_j, u_i)\mathbb{E}\left[\frac{f_{t-}(u_i)\lambda_t V_t}{V_t + \lambda_t \bar{\kappa}}\right] + C_4(t, u_j, u_i)\mathbb{E}\left[\frac{f_{t-}(u_j)\lambda_t V_t}{V_t + \lambda_t \bar{\kappa}}\right] \\ & + C_5(t, u_j, u_i)\mathbb{E}\left[\frac{f_{t-}(u_i) f_{t-}(u_j)\lambda_t^2}{V_t + \lambda_t \bar{\kappa}}\right] \zeta^j(\mathrm{d}u_i)\zeta^j(\mathrm{d}u_j)\mathrm{d}t, \end{aligned}$$

where $\bar{\kappa}$ constant and $f_{t-}(u_j) = e^{\phi_{T-t}(u_j,0,0) + u_j X_{t-} + \psi_{T-t}(u_j,0,0) V_{t-} + \chi_{T-t}(u_j,0,0)\lambda_{t-}}$ with A_i, B_i, C_i deterministic functions of t, u_i, u_j and of the model's parameters.

Applications to energy markets

In progress

- In energy markets variance swaps are typically defined on Futures contracts $\rightarrow$ To hedge a variance swap in energy markets, we need to consider a basket of contingent claims written on Futures.
- Let $F(t, T_1)$ be the t-price of a Futures contract with maturity $T_1 > 0$, we can assume that $\log F(t, T_1)$ follows the same dynamics of $X_t \rightarrow$ The previous analysis on the hedging strategies remains the same also in the case of options written on Futures.

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1. Calibrate our model parameters on the prices of options written on Futures for a chosen commodity (e.g. crude oil).

X_0	γ	δ	ρ	V_0	α_V	β_V	σ_V	λ_0	α_λ	β_λ	ζ
4.605	-0.0099	0.0296	-0.7163	0.0242	0.0175	6.7272	0.6872	7.2448	3.8283	8.8334	0.34

Data on crude oil, Brignone *et al.* 2023

2. Compute numerically the hedging strategy.
   - Computation of $\mathbb{E}[e^{uX}|\mathcal{F}_t] \rightarrow$ computation of the solution of the Riccati system (uncoupled system of complex ODEs) $\rightarrow$ numerical complex ODEs solver.
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Applications in energy markets

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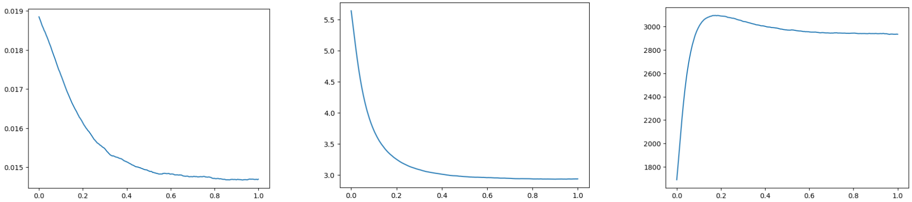

Figure: Expectations of $\frac{V_t^2}{V_t+\lambda_t\bar{\kappa}}$, $\frac{\lambda_t V_t}{V_t+\lambda_t\bar{\kappa}}$, $\frac{\lambda_t^2}{V_t+\lambda_t\bar{\kappa}}$, respectively, for $t \in [0, 1]$.

Future numerical investigations

We also aim at investigating the following questions:

- If a semi-static strategy is actually better than a fully dynamic one, in our specific model.
- Optimal selection of static hedging assets: *How many assets are enough to obtain a "reasonably small" hedging error? And which asset in the market should be chosen?*

Thank you!

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References I

[1] Y. Aït-Sahalia, J. Cacho-Diaz, and R. J. Laeven, "Modeling financial contagion using mutually exciting jump processes," *Journal of Financial Economics*, vol. 117, no. 3, pp. 585–606, 2015.

[2] L. B. Andersen and V. V. Piterbarg, "Moment Explosions in Stochastic Volatility Models," *Finance and Stochastics*, vol. 11, no. 1, pp. 29–50, 2007.

[3] P. Baldi, *Stochastic Calculus: An Introduction Through Theory and Exercises*. Springer International Publishing, 2017.

[4] M. Beiglböck, P. Henry-Labordère, and F. Penkner, "Model-independent bounds for option prices: A mass transport approach.," *Finance and Stochastics*, vol. 17, no. 3, pp. 477–501, 2013.

[5] M. Brachetta, G. Callegaro, C. Ceci, and C. Sgarra, "Optimal reinsurance via BSDEs in a partially observable contagion model with jump clusters.," *Finance and Stochastics*, 2023.

[6] R. Brignone, L. Gonzato, and C. Sgarra, "Commodity Asian option pricing and simulation in a 4-factor model with jump clusters," *Annals of Operations Research*, pp. 1–32, 2023.

[7] P. Carr, "Semi-static hedging of barrier options under Poisson jumps," *International Journal of Theoretical and Applied Finance*, vol. 14, no. 07, pp. 1091–1111, 2011.

[8] P. Carr and A. Corso, "Covariance contracting for commodities," *Energy Risk*, vol. April, pp. 42–45, 2001.

[9] P. Carr and D. Madan, "Towards a theory of volatility trading," in *Option pricing, interest rates and risk management, Handbooks Mathematical Finance*, E. Jouini, J. Cvitanic, and M. Musiela, Eds., Cambridge: Cambridge University Press, 2001, pp. 458–476.

[10] A. Dassios and H. Zhao, "A dynamic contagion process," *Advances in Applied Probability*, vol. 43, no. 3, pp. 814–846, 2011.

[11] P. Di Tella, M. Haubold, and M. Keller-Ressel, "Semi-static variance-optimal hedging in stochastic volatility models with Fourier representation," *Journal of Applied Probability*, vol. 56, no. 3, pp. 787–809, 2019.

[12] P. Di Tella, M. Haubold, and M. Keller-Ressel, "Semistatic and sparse variance-optimal hedging," *Mathematical Finance*, vol. 30, no. 2, pp. 403–425, 2020.

[13] G. G. Drimus, "Options on realized variance by transform methods: A non-affine stochastic volatility model," *Quantitative Finance*, vol. 12, no. 11, pp. 1679–1694, 2012.

[14] D. Duffie, D. Filipović, and W. Schachermayer, "Affine processes and applications in finance," *The Annals of Applied Probability*, vol. 13, no. 3, pp. 984–1053, 2003.

[15] E. Eberlein and J. Kallsen, *Mathematical Finance*. Cham, Switzerland: Springer Finanace, 2019.

References II

[16] V. Filimonov, D. Bicchetti, N. Maystre, and D. Sornette, "Quantification of the high level of endogeneity and of structural regime shifts in commodity markets," *Journal of international Money and Finance*, vol. 42, pp. 174–192, 2014.

[17] H. Föllmer and D. Sondermann, "Hedging of non-reduntant contingent claims," in *Contributions to mathematical economics*, W. Hildenbrand and A. MasColell, Eds., North Holland, The Netherlands: Elsevier Science Publishing, 1986, pp. 205–223.

[18] D. Hainaut and F. Moraux, "Hedging of options in the presence of jump clustering," *Journal of Computational Finance*, vol. 22, no. 3, pp. 1–35, 2018.

[19] A. G. Hawkes, "Spectra of some self-exciting and mutually exciting point processes," *Biometrika*, vol. 58, no. 1, pp. 83–90, 1971.

[20] S.-W. He, J.-G. Wang, and J.-a. Yan, *Semimartingale Theory and Stochastic Calculus*. Routledge, 1992.

[21] R. Herrera and N. González, "The modeling and forecasting of extreme events in electricity spot markets," *International Journal of Forecasting*, vol. 30, no. 3, pp. 477–490, 2014.

[22] S. L. Heston, "A closed-form solution for options with stochastic volatility with applications to bond and currency options," *The Review of Financial Studies*, vol. 6, no. 2, pp. 327–343, 1993.

[23] F. Hubalek, J. Kallsen, and L. Krawczyk, "Variance-optimal hedging for processes with stationary independent increments," *The Annals of Applied Probability*, vol. 16, pp. 853–885, 2006.

[24] J. Jacod, "Multivariate point processes: Predictable projection, Radon-Nikodym derivatives, representation of martingales," *Z. Wahrscheinlichkeitstheorie verw Gebiete*, vol. 31, pp. 235–253, 1975.

[25] J. Jacod and A. Shiryaev, *Limit theorems for stochastic processes*. Springer Science & Business Media, 2013, vol. 288.

[26] J. Kallsen and A. Pauwels, "Variance-optimal hedging in general affine stochastic volatility models," *Advances in Applied Probability*, vol. 42, no. 1, pp. 83–105, 2010.

[27] M. Keller-Ressel, "Moment explosions and long-term behavior of affine stochastic volatility models," *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics*, vol. 21, no. 1, pp. 73–98, 2011.

[28] M. Keller-Ressel and M. Eberhard, "Exponential moments of affine processes," *The Annals of Applied Probability*, vol. 25, no. 2, pp. 714–752, 2015.

[29] H. Kunita and S. Watanabe, "On square integrable martingales," *Nagoya Mathematical Journal*, vol. 30, pp. 209–245, 1967.

[30] W. Liu and S.-P. Zhu, "Pricing variance swaps under the Hawkes jump-diffusion process," *Journal of Futures Markets*, vol. 39, no. 6, pp. 635–655, 2019.

[31] A. Neuberger, "The log contract," *Journal of Portfolio Management*, vol. 20, no. 2, pp. 74–80, 1994.

References III

[32] M. Prokopczuk, L. Symeonidis, and C. Wese Simen, "Variance risk in commodity markets," *Journal of Banking and Finance*, vol. 81, pp. 136–149, 2017.

[33] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, 1999, vol. 68.

[34] A. Swishchuk, "Variance and volatility swaps in energy markets," *The Journal of Energy Markets*, vol. 6, pp. 33–49, 2013.

[35] A. Trolle and E. S. Schwartz, "Variance risk premia in energy commodities," *Journal of Derivatives*, vol. 17, pp. 15–32, 2010.