# Optimal Investment Strategies under the Relative Performance in Jump-Diffusion Markets

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# Outline

# Introduction

- Our problem: Determine the optimal investment strategies of one agent and (a large group of agents) having CRRA utilities under relative performance concerns.
- There is a strategic interaction between the agent and the group which is caused by their relative performances.
- We solve two classical Merton problems to determine what the group does and the agent's optimal strategy relative to the group's performance.
- There are some studies which revealed the significance of the relative performance concerns on human behaviors, see for instance [3] and [5].
- In our framework, the stock price is defined on jump-diffusion markets where we both consider the upward and downward jumps in the numerical observations.

- Basak and Makarov [1] work on a portfolio choice problem with a competition between two risk-averse portfolio managers where the stock follows geometric Brownian motion.
- Espinosa and Touzi [4] consider $n$ interacting agents maximizing the difference between an agent's wealth and average wealth of other agents' wealth.
- Lacker and Zariphopoulou [7] work on an optimal portfolio management problem for interacting $n$ interacting agents with both CARA and CRRA utilities. They obtain equilibrium strategies for both $n$ agents and mean-field games.
- Lacker and Soret [6] extend the CRRA model developed by Lacker and Zariphopoulou [7] for a continuous time-dependent problem including relative performances.
- Reis and Platonov [8] develop equilibrium strategies using forward utilities of CARA type.
- Bäuerle and Göll [2] obtain Nash equilibrium investment strategies under the strategic interactions of $n$ agents investigating the influence of the price impact on the equilibrium.

- Our work is both an extension and a special case of the paper Lacker and Zariphopoulou [7] in the following directions:

  1. We simplify their mean-field equilibrium results by considering only one agent and a group of agents.
  2. In our framework, the stock price is followed in a jump-diffusion market as follows:

     $$dS_t = S_t\Big(\mu dt + \nu dW_t + \sigma dB_t - \varepsilon dN_t - \tilde{\varepsilon} d\tilde{N}_t\Big)$$

     with constant parameters $\mu, \nu, \sigma > 0$; the constant jump sizes $\varepsilon, \tilde{\varepsilon}$; two independent standard Brownian motions $W_t$ and $B_t$ and two independent Poisson processes $N_t$ and $\tilde{N}_t$.
  3. Moreover, we assume that the agents in the market are homogeneous in their risk-aversions and relative performances as a special case.
  4. The model is an extension in the sense that we allow for jumps in the market. However, it is a special case because we only consider the mean-field equilibrium, not the $n$-agent equilibrium.

# Framework

- We define a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a sample space $\Omega$, a filtration $\mathcal{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ and a probability measure $\mathbb{P}$.
- The agent and the group trade using self-financing strategies $\pi_t$ and $\hat{\pi}_t$ which represent the optimal proportional allocations in the portfolio, respectively.
- We assume that the financial market includes one riskless asset and one risky asset for the agent to invest at times $t \in [0, T]$. The price process of the riskless asset follows the dynamics,

$$dS_t^0 = rS_t^0 dt$$

with a constant interest rate $r \geq 0$. We assume for simplicity that $r = 0$.

- The process of the risky asset is generated in a jump-diffusion market as follows:

$$dS_t = S_t\Big(\mu dt + \nu dW_t + \sigma dB_t - \varepsilon dN_t - \tilde{\varepsilon} d\tilde{N}_t\Big)$$

- Along with the agents, there exists a group that invests in a different risky asset, which is also generated in a jump-diffusion market as follows:

$$d\hat{S}_t = \hat{S}_t\Big(\hat{\mu} dt + \hat{\sigma} dB_t - \hat{\varepsilon} dN_t\Big)$$

  with constant parameters $\hat{\mu}, \hat{\sigma} > 0$, and a constant jump size $\hat{\varepsilon} \in \mathbb{R}$.

- Let $X$ and $Y$ denote the wealth of the agent and the group. We suppose that they are generated by the dynamics,

$$dX_t = \pi_t X_t \Big(\mu dt + \nu dW_t + \sigma dB_t - \varepsilon dN_t - \tilde{\varepsilon} d\tilde{N}_t\Big),$$
$$X_0 = x_0; \tag{1}$$
$$dY_t = \hat{\pi}_t Y_t \Big(\hat{\mu} dt + \hat{\sigma} dB_t - \hat{\varepsilon} dN_t\Big),$$
$$Y_0 = y_0. \tag{2}$$

- In this setup,

  1. $\mu, \hat{\mu}$ are the constant drift parameters;
  2. $\nu, \sigma, \hat{\sigma} > 0$ are the constant volatility parameters;
  3. $W_t$ and $B_t$ are independent standard Brownian motions; $W$ is an idiosyncratic noise and $B$ is the common noise;
  4. $\varepsilon, \tilde{\varepsilon}, \hat{\varepsilon}$ are the constant jump sizes;
  5. $N_t$ and $\tilde{N}_t$ are two independent Poisson processes with the constant intensities $\lambda$ and $\tilde{\lambda}$, respectively;
  6. the allocations $\pi$ and $\hat{\pi}$ determine the proportion of the agent's wealth and the group's wealth, respectively.

# Optimal control problem for the agent

- The utility function of the agent:

$$U(X,Y) = \frac{1}{1-\gamma}\left(XY^{-\theta}\right)^{1-\gamma} \tag{3}$$

where $\gamma > 0$, $\gamma \neq 1$ is the risk-aversion of the agent and $0 \leq \theta \leq 1$ specifies the influence from the group's performance through $Y$ on the utility of the agent.

- Some comments on the utility function:
  - When $\theta = 0$, the group's performance does not impact the agent's performance.
  - When $\theta = 1$, the utility function (3) is

$$U(X,Y) = \frac{1}{1-\gamma}\left(\frac{X}{Y}\right)^{1-\gamma},$$

such that the utility is gained from a direct relative performance between the agent and the group.

- The stochastic optimal control problem of the agent:

$$
\begin{aligned}
w(x,y,t) &= \sup_{\pi\in\mathcal{A}} \mathbb{E}\left[U(X(T),Y(T)) \mid X_0 = x, Y_0 = y\right] \\
&= \sup_{\pi\in\mathcal{A}} \mathbb{E}\left[\frac{1}{1-\gamma}\left(X(T)Y^{-\theta}(T)\right)^{1-\gamma} \middle| X_0 = x, Y_0 = y\right], \qquad (4)
\end{aligned}
$$

where $\mathcal{A}$ is the set of strategies, called admissible, such that (1) has a solution and (4) is finite.

- When we solve the problem, we always solve for $\pi$, which is constant, such that the conditions for admissibility hold trivially.
- The Hamilton-Jacobi-Bellman (HJB) equation:

$$
\begin{aligned}
0 =& w_t + \hat{\mu} y \hat{\pi} w_y + \frac{1}{2}\hat{\sigma}^2 y^2 \hat{\pi}^2 w_{yy} + \sup_{\pi\in\mathcal{A}} \Big\{ \pi\left(\mu x w_x + \sigma\hat{\sigma} x y \hat{\pi} w_{xy}\right) + \frac{1}{2}\left(v^2+\sigma^2\right)\pi^2 x^2 w_{xx} \\
&+ \lambda\Big(w(x-\pi x\varepsilon, y-\hat{\pi} y\hat{\varepsilon}, t) - w(x,y,t)\Big) + \tilde{\lambda}\Big(w(x-\pi x\tilde{\varepsilon}, y, t) - w(x,y,t)\Big)\Big\} \qquad (5)
\end{aligned}
$$

for $(x,y,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \times [0,T]$ with the final condition

$$
w(x,y,T) = \frac{1}{1-\gamma}\left(xy^{-\theta}\right)^{1-\gamma}.
$$

- The ansatz $w(x,y,t) = g(t)\frac{1}{1-\gamma}\left(xy^{-\theta}\right)^{1-\gamma}$ yields the ODE

$$g'(t) + \tilde{\rho} g(t) = 0$$

where

$$\tilde{\rho} = -\hat{\mu}\hat{\pi}\theta(1-\gamma) - \frac{1}{2}\hat{\sigma}^2\hat{\pi}^2\theta(\theta\gamma - \theta - 1)(1-\gamma)$$

$$+ \sup_{\pi\in\mathcal{A}}\left\{\mu\pi(1-\gamma) - \sigma\hat{\sigma}\hat{\pi}\theta(1-\gamma)^2\pi - \frac{1}{2}(\nu^2+\sigma^2)\pi^2\gamma(1-\gamma)\right.$$

$$\left. +\lambda\left(\left((1-\pi\varepsilon)(1-\hat{\pi}\hat{\varepsilon})^{-\theta}\right)^{1-\gamma} - 1\right) + \tilde{\lambda}\left((1-\pi\tilde{\varepsilon})^{1-\gamma} - 1\right)\right\}.$$

- The resulting equation for $\pi$:

$$\mu - \sigma\hat{\sigma}\hat{\pi}\theta(1-\gamma) - (\nu^2+\sigma^2)\pi\gamma - \lambda(1-\pi\varepsilon)^{-\gamma}\varepsilon(1-\hat{\pi}\hat{\epsilon})^{-\theta(1-\gamma)} - \tilde{\lambda}(1-\pi\tilde{\varepsilon})^{-\gamma}\tilde{\varepsilon} = 0. \tag{6}$$

- Considering the equation (6) as a function of $\pi$, i.e., $f(\pi) = 0$, the existence and uniqueness of the solution can be proved depending on the parameters.
- Therefore, we guarantee the existence and uniqueness of a solution if the parameters are set, and we avoid the discontinuity points.

**Remark**

*While the above result is about the general uniqueness and existence results, it is also worth highlighting our particular cases. In all our numerical examples, in each case, we put specific parameters equal to* $0$ *to linearize the equation (6) and isolate* $\pi$*. Therefore, we do not need to check the general theory in these cases because we find explicitly a single solution.*

# Optimal strategies of the agent:

Remember that the resulting equation for $\pi$ is

$$\mu - \sigma\hat{\sigma}\hat{\pi}\theta(1-\gamma) - (\nu^2+\sigma^2)\pi\gamma - \lambda(1-\pi\varepsilon)^{-\gamma}\varepsilon(1-\hat{\pi}\hat{\epsilon})^{-\theta(1-\gamma)} - \tilde{\lambda}(1-\pi\tilde{\varepsilon})^{-\gamma}\tilde{\varepsilon} = 0.$$

1. The pure diffusion case ($\varepsilon = \tilde{\varepsilon} = 0$):

$$\pi^* = \frac{\frac{1}{\gamma}\mu - \sigma\hat{\sigma}\hat{\pi}\theta(\frac{1}{\gamma}-1)}{v^2+\sigma^2} \tag{7}$$

where $\frac{1}{\gamma}$ is defined as the risk-tolerance parameter of the agent.

- The mean field equilibrium strategy builds on the idea that the group above consists of many agents, each corresponding to the agent above. Thus, the agent's performance is measured relative to a group of many agents to which she belongs.
- By many, we mean, in principle, infinitely many such that the influence on the group's performance from every single agent is negligible.
- This means that, in the limit, the agent's performance does not influence the group's performance. Still, the group's performance influences the performance of the individual via the parameter $\theta$.

- We discuss this result in relation to certain results achieved by Lacker and Zariphopoulou [7] where they obtain a Mean Field Equilibrium strategy under relative performances.

## Lemma

*Assuming that the agent has the optimal strategy $\pi$ given by (7), we recover the Mean Field Equilibrium strategy in Lacker and Zariphopoulou [7].*

## Proof.

- The Mean Field Equilibrium strategy formula in Lacker and Zariphopoulou [7] reads

$$\pi^* = \delta \cdot \frac{\mu}{\sigma^2 + v^2} - \theta(\delta - 1)\frac{\sigma}{\sigma^2 + v^2} \cdot \frac{\varphi}{1 + \psi}, \tag{8}$$

where the constants $\varphi$ and $\psi$ are defined as

$$\varphi := \mathbb{E}\left[\delta \frac{\mu\sigma}{\sigma^2 + \nu^2}\right] \quad \text{and} \quad \psi := \mathbb{E}\left[\theta(\delta - 1)\frac{\sigma^2}{\sigma^2 + \nu^2}\right],$$

the expectations are essentially averages over individuals in their mean-field approach and $\delta = 1/\gamma$.

## Proof (Cont.)

- Multiply both sides of (7) by $\sigma$ to obtain

$$\sigma\pi^* = \frac{\frac{1}{\gamma}\mu\sigma}{v^2+\sigma^2} - \frac{\sigma^2\hat{\sigma}\hat{\pi}\theta(\frac{1}{\gamma}-1)}{v^2+\sigma^2}. \tag{9}$$

- We now take the average over all group agents on both sides. This operation is essentially the 'mean-field' part of the 'mean-field equilibrium'. On the right-hand side, the expressions $\varphi$ and $\psi$ appear in the following expression

$$\mathbb{E}\left[\frac{\frac{1}{\gamma}\mu\sigma}{\nu^2+\sigma^2}\right] - \mathbb{E}\left[\frac{\sigma^2\hat{\sigma}\hat{\pi}\theta(\frac{1}{\gamma}-1)}{\nu^2+\sigma^2}\right] = \psi + \hat{\sigma}\hat{\pi}\varphi.$$

- We now set the average over products of volatilities and investment proportions, $\mathbb{E}[\sigma\pi^*]$, equal to the corresponding group product $\hat{\sigma}\hat{\pi}$,

$$\hat{\sigma}\hat{\pi} = \mathbb{E}[\sigma\pi^*].$$

  This operation is essentially the 'equilibrium' part of the 'mean-field equilibrium'.
- Now, equating two sides and solving for $\hat{\sigma}\hat{\pi}$ gives us

$$\hat{\sigma}\hat{\pi} = \frac{\psi}{1+\varphi}.$$

- Plugging that back into (7) gives exactly Equation (8) in Lacker and Zariphopoulou [7].

- Single stock case:

**Lemma**

*Assume that* $\hat{\mu} = \mu, \hat{\sigma} = \sigma$ *and* $v = 0$. *Then, the optimal strategy reads as*

$$\pi^* = \frac{\mu}{\sigma^2}\left(\frac{1}{\gamma} - \frac{\mathbb{E}\left[\frac{1}{\gamma}\right]}{1 + \mathbb{E}\left[\theta\left(\frac{1}{\gamma} - 1\right)\right]}\theta\left(\frac{1}{\gamma} - 1\right)\right). \tag{10}$$

**Proof.**

- When $\hat{\mu} = \mu, \hat{\sigma} = \sigma$ and $v = 0$, (7) simplifies to

$$\pi^* = \frac{1}{\gamma}\frac{\mu}{\sigma^2} - \hat{\pi}\theta\left(\frac{1}{\gamma} - 1\right). \tag{11}$$

## Proof (Cont.)

- Averaging over both sides gives

$$\hat{\pi} = \mathbb{E}\left[\frac{1}{\gamma}\right]\frac{\mu}{\sigma^2} - \hat{\pi}\mathbb{E}\left[\theta\left(\frac{1}{\gamma} - 1\right)\right].$$

- Solved for $\hat{\pi}$ gives

$$\hat{\pi} = \frac{\mathbb{E}\left[\frac{1}{\gamma}\right]\frac{\mu}{\sigma^2}}{1 + \mathbb{E}\left[\theta\left(\frac{1}{\gamma} - 1\right)\right]}.$$

- Then, plugged into $\pi^*$ in (11) gives us (10).

□


Figure 1: Optimal strategy $\pi^*$ with the common parameters $\mu = 5$, $\sigma = 1$ in the single stock case. Left: $\pi^*$ in Lacker and Zariphopoulou [7] versus $\bar{\theta} = \mathbb{E}[\theta]$ and $\delta = \frac{1}{\gamma}$ with $\theta = \frac{3}{4}$ and $\bar{\delta} = \mathbb{E}\left[\frac{1}{\gamma}\right] = 2$. Right: $\pi^*$ in (10) versus $\theta$ and $\frac{1}{\gamma}$ with $\mathbb{E}[\theta] = \frac{3}{4}$ and $\mathbb{E}\left[\frac{1}{\gamma}\right] = 2$.

- Note that we assume $\theta$ and $\frac{1}{\gamma}$ are uncorrelated to generate the above results.
- When $\theta = 0$, then $\pi^*$ increases linearly.
- When $\theta = 1$, $\pi^*$ is independent of $\gamma$.
- When $\gamma > 1$, $\pi^*$ increases in $\theta$.

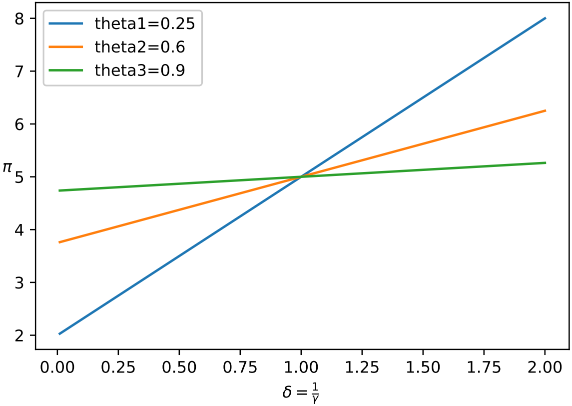


Figure 2: Optimal investment strategy versus $\delta = \frac{1}{\gamma}$ for different $\theta$ values.

- Homogeneous agents case:
  The agents are homogeneous in their risk-aversions and relative performances, i.e. $\mathbb{E}\left[\frac{1}{\gamma}\right] = \frac{1}{\gamma}$ and $\mathbb{E}[\theta] = \theta$.

  By this assumption, (10) turns out to be

$$\pi^* = \frac{\mu}{\sigma^2\gamma}\left(\frac{1}{1+\theta\left(\frac{1}{\gamma}-1\right)}\right). \tag{12}$$

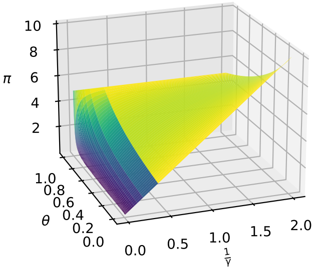


Figure 3: Optimal investment strategy versus $\theta$ and $\gamma$ with the parameters $\mu = 0.2$, $\sigma = 0.2$ in the homogeneous agents case for a single stock.

- We check the optimal $\pi^*$ values in the right picture of Figure 1 and in Figure 3 for $\theta = 0.75$ and $\frac{1}{\gamma} = 2$. It is found as $5.7143$ in both figures.

## Corollary

*If the agents are homogeneous in their preferences, the optimal strategy of the representative agent is given as different depending on the boundary values of the influence constant $\theta$.*

1. *When the group $Y$ has no influence on the agent $X$, that is $\theta = 0$, we have*

$$\pi^* = \frac{\mu}{\sigma^2 \gamma}$$

*which is also the result under the pure jump diffusion case for the group. However, in that case, we should have the risk aversion degree of the group appearing in the formula instead of $\gamma$ since it would represent the strategy only for the group.*

2. *When the group $Y$ has full impact on the agent $X$, that is $\theta = 1$, (12) returns to be*

$$\pi^* = \frac{\mu}{\sigma^2}.$$

*In this case, we observe that the risk aversion degree of the agent does not show up in the optimal strategy.*

1. The pure jump case ($\sigma = \nu = 0$):

We consider an assumption such that the diffusion constant parameters are equal to zero to isolate the optimal strategy $\pi^*$ by (6).

- No individual jump risk:

  We consider the case where there is no individual jump risk, i.e., $\tilde{\varepsilon} = 0$. Hence, (6) reduces to

$$\mu - \lambda(1 - \pi\varepsilon)^{-\gamma}\varepsilon(1 - \hat{\pi}\hat{\varepsilon})^{-\theta(1-\gamma)} = 0.$$

  Hence, we obtain the strategy as

$$\pi^* = \frac{1}{\varepsilon} - \left(\frac{\mu}{\lambda}\right)^{-\frac{1}{\gamma}} \frac{(1 - \hat{\pi}\hat{\varepsilon})^{\theta\left(1-\frac{1}{\gamma}\right)}}{\varepsilon^{\left(1-\frac{1}{\gamma}\right)}}. \tag{13}$$

- Single stock case:

**Lemma**

*Assume that $\hat{\mu} = \mu$ and $\varepsilon = \hat{\varepsilon}$. Then, the optimal strategy without an individual jump risk reads as*

$$\hat{\pi} = \mathbb{E}\left(\frac{1}{\varepsilon} - \left(\frac{\mu}{\lambda}\right)^{-\frac{1}{\gamma}} \frac{(1-\hat{\pi}\varepsilon)^{\theta\left(1-\frac{1}{\gamma}\right)}}{\varepsilon^{\left(1-\frac{1}{\gamma}\right)}}\right). \tag{14}$$

**Proof.**

With the assumptions of $\hat{\mu} = \mu$ and $\varepsilon = \hat{\varepsilon}$ and averaging both sides, we obtain (14). □

- Homogeneous agents case:
  We assume that the agents are homogeneous in the single stock.
  Then, the strategy (14) returns to be

$$\pi^* = \frac{1}{\varepsilon} - \left(\frac{\mu}{\lambda}\right)^{-\frac{1}{\gamma}} \frac{(1-\pi^*\varepsilon)^{\theta\left(1-\frac{1}{\gamma}\right)}}{\varepsilon^{\left(1-\frac{1}{\gamma}\right)}} \tag{15}$$

for a group of homogeneous agents where the average over agents disappears and $\pi^* = \hat{\pi}$.
By some direct calculations in (15), we isolate $\pi$ as

$$\pi^* = \frac{1}{\varepsilon} - \frac{\left(\frac{\lambda\varepsilon}{\mu}\right)^{\frac{1}{\gamma}\left(\frac{1}{1+\theta\left(\frac{1}{\gamma}-1\right)}\right)}}{\varepsilon}. \tag{16}$$

## Corollary

*The optimal strategy (16) receives distinct values on the boundary values of $\theta$.*

1. *When the group performance does not affect the agent's performance, that is $\theta = 0$, the optimal strategy is given by*

$$\pi^* = \frac{1}{\varepsilon} - \frac{\left(\frac{\lambda\varepsilon}{\mu}\right)^{\frac{1}{\gamma}}}{\varepsilon}.$$

2. *When the agent's performance is completely dependent on the group's performance, that is $\theta = 1$, the optimal strategy is found as*

$$\pi^* = \frac{1}{\varepsilon} - \frac{\lambda}{\mu}.$$

- Parameter choice for the numerical illustrations:

  Now, we display the optimal strategies under this case. The wealth process X of the agent have the same expected return and uncertainty in terms of the local variance as in the pure diffusion case. Then, we have the following two equations with three unknowns:

$$\begin{aligned} \mu - \varepsilon\lambda &= 0.2 \\ \varepsilon^2\lambda &= 0.04. \end{aligned}$$

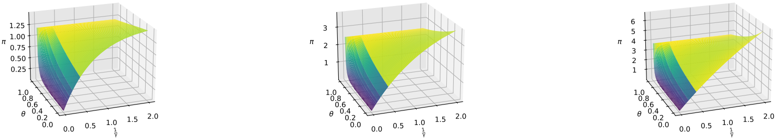


Figure 4: Optimal strategies versus $\theta$ and $\frac{1}{\gamma}$ for $\mu = \{0.263245, 0.4, 0.83245\}$, $\lambda = \{0.1, 1, 10\}$, $\varepsilon = \{0.63245, 0.2, 0.063245\}$ in a pure jump model with downward jumps when $\tilde{\varepsilon} = 0$.

- The parameter $\lambda$ is the frequency of the crashes in the market.
- When the downward jumps are larger, the investor is avoiding to invest more as $\pi^*$ goes down.
- When $\lambda \to \infty$, $\varepsilon \to 0$ and so that the optimal strategy approaches to the strategy obtained by (12) in the pure diffusion case of the homogeneous agents.

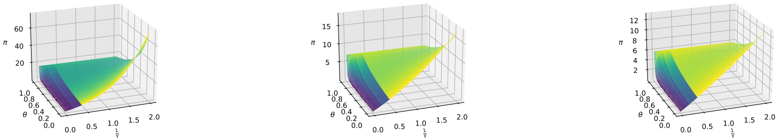


Figure 5: Optimal strategies versus $\theta$ and $\frac{1}{\gamma}$ for $\mu = \{-0.08284, -0.43245, -1.0648\}$, $\lambda = \{2, 10, 40\}$, $\varepsilon = \{-0.14142, -0.063245, -0.03162\}$ in a pure jump model with upward jumps when $\tilde{\varepsilon} = 0$.

- When $\lambda$ gets larger, $\pi^*$ converges to the pure-diffusion case.
- The model produces negative $\varepsilon$ and positive $\mu$ values when $\lambda = 0.1$ which results in no solution.
- When $\lambda = 1$, then we find that $\mu = 0$ under the zero interest rate assumption. Then, there is an opportunity to win when there are stocks.
- Therefore, when $\lambda \leq 1$, the model contains arbitrage and the portfolio optimization problem has no solution.
- The investor gets a positive return on the stock, which creates even more arbitrage between the jumps because she borrows the money in the bank, buys stocks and earns an extra return $\mu$, larger than $0$.

- Homogeneous jump sizes:

When $\varepsilon = \tilde{\varepsilon}$, the jump sizes become homogeneous..
In this case, (6) deduces to

$$\mu - \lambda(1-\pi\varepsilon)^{-\gamma}\varepsilon(1-\hat{\pi}\hat{\varepsilon})^{-\theta(1-\gamma)} - \tilde{\lambda}(1-\pi\varepsilon)^{-\gamma}\varepsilon = 0.$$

Thus, the optimal strategy reads as

$$\pi^* = \frac{1}{\varepsilon} - \frac{\left(\frac{\mu}{\left(\lambda(1-\hat{\pi}\hat{\varepsilon})^{-\theta(1-\gamma)}+\tilde{\lambda}\right)\varepsilon}\right)^{-\frac{1}{\gamma}}}{\varepsilon}. \tag{17}$$

**Lemma**

*(Single stock case) Assume that $\hat{\mu} = \mu$, $\varepsilon = \tilde{\varepsilon} = \hat{\varepsilon}$ and $\lambda = \tilde{\lambda}$. Then, the optimal strategy can be obtained as*

$$\hat{\pi} = \mathbb{E}\left(\frac{1}{\varepsilon} - \frac{1-\left(\frac{\mu}{\lambda\varepsilon\left((1-\hat{\pi}\varepsilon)^{-\theta(1-\gamma)}+1\right)}\right)^{-\frac{1}{\gamma}}}{\varepsilon}\right) \tag{18}$$

*where we also take the average of both sides.*

- Homogeneous agents case:

  In this case, the strategy is

$$\pi^* = \frac{1}{\varepsilon} - \frac{1 - \left(\frac{\mu}{\lambda\varepsilon\left((1-\pi^*\varepsilon)^{-\theta(1-\gamma)}+1\right)}\right)^{-\frac{1}{\gamma}}}{\varepsilon}. \tag{19}$$

  By some straightforward calculations, we have

$$\frac{\mu}{\lambda\varepsilon}(1-\pi^*\varepsilon)^{\gamma} - (1-\pi^*\varepsilon)^{-\theta(1-\gamma)} = 1. \tag{20}$$

## Corollary

*The optimal strategy $\pi^*$ can get different values for any $\theta$ values.*

1. *When $\theta = 0$ in (20), the optimal strategy $\pi$ is given by*

$$\pi^* = \frac{1}{\varepsilon} - \frac{\left(\frac{\mu}{2\lambda\varepsilon}\right)^{-\frac{1}{\gamma}}}{\varepsilon}. \tag{21}$$

2. *When $\theta = 1$ in (20), we obtain an equation with different powers of $(1-\pi^*\varepsilon)$. Therefore, we cannot isolate $\pi^*$ explicitly in this case.*

- Independent individual jump risk:

When $\varepsilon = 0$ in (1), there is no correlation between the agent and the group. In this case, (6) turns out to be

$$\mu - \tilde{\lambda}(1 - \pi\tilde{\varepsilon})^{-\gamma}\tilde{\varepsilon} = 0.$$

Then, the optimal strategy is obtained as

$$\pi^* = \frac{1 - \left(\frac{\mu}{\tilde{\lambda}\tilde{\varepsilon}}\right)^{-\frac{1}{\gamma}}}{\tilde{\varepsilon}}. \tag{22}$$

**Remark**

*When $\sigma = 0$ and $\varepsilon = 0$, we have a jump-diffusion case but we do not have joint risk between the agent and the group. Therefore, $\theta$ disappears in the optimal strategy formula, e.g. in (22), since it does not matter in this case. $\theta$ only appears together with joint risk, that is, when $\sigma = \hat{\sigma} > 0$ or $\varepsilon = \hat{\varepsilon} > 0$.*

## Remark

*When the agent does not care about the group's performance, that is when $\theta = 0$, the risk-aversion degree of the agent, which is $\gamma$, always appears on the optimal strategy formula. This result makes sense because the degree of the risk aversion of the agent determines the strategy playing an important role. When $\theta = 1$, no $\gamma$ appears on the formulas.*

- The utility function of the group:

$$V(Y) = \frac{1}{1-\eta} Y^{1-\eta}$$

where $\eta > 0$, $\eta \neq 1$ is the risk-aversion of the group.

- The optimization problem of the group:

$$u(y,t) = \sup_{\hat{\pi} \in \hat{\mathcal{A}}} \mathbb{E}[V(Y)] = \sup_{\hat{\pi} \in \hat{\mathcal{A}}} \mathbb{E}[V(y) \mid Y_0 = y] \tag{23}$$

where $\hat{\mathcal{A}}$ is the set of admissible strategies

- The Hamilton-Jacobi-Bellman (HJB) equation for the value function (23):

$$u_t + \sup_{\hat{\pi} \in \mathcal{A}} \left\{ \hat{\mu}\hat{\pi}\ y u_y + \frac{1}{2}\hat{\sigma}^2\hat{\pi}^2 y^2 u_{yy} + \lambda\Big(u(y - \hat{\pi} y \hat{\varepsilon}, t) - u(y,t)\Big) \right\} = 0 \tag{24}$$

for $(y, t) \in \mathbb{R}_+ \times [0, T]$ with the terminal condition

$$u(y, T) = \frac{1}{1-\eta} y^{1-\eta}.$$

- The resulting equation for $\hat{\pi}$:

$$\hat{\mu} - \hat{\sigma}^2 \hat{\pi}\eta - \lambda\hat{\varepsilon}(1 - \hat{\pi}\hat{\varepsilon})^{-\eta} = 0. \tag{25}$$

# Optimal strategies of the group:

- The pure diffusion case ($\hat{\varepsilon} = 0$):

$$\hat{\pi}^* = \frac{\hat{\mu}}{\hat{\sigma}^2 \eta}, \tag{26}$$

which is the classical Merton optimal strategy.
Note that (26) can be used for a small benchmark portfolio to make a comparison with a market benchmark portfolio.

- The pure jump case ($\hat{\sigma} = 0$):

$$\hat{\pi}^* = \frac{1}{\hat{\varepsilon}} - \frac{\left(\frac{\hat{\mu}}{\lambda \hat{\varepsilon}}\right)^{-\frac{1}{\eta}}}{\hat{\varepsilon}}. \tag{27}$$

- Small benchmark portfolio without individual jump risk:

  In this case, we insert (27) into (13) to get

$$\pi^* = \frac{1}{\varepsilon} - \left(\frac{\mu}{\lambda}\right)^{-\frac{1}{\gamma}} \frac{\left(\frac{\hat{\mu}}{\lambda\hat{\varepsilon}}\right)^{\frac{\theta}{\eta}\left(\frac{1}{\gamma}-1\right)}}{\varepsilon^{\left(1-\frac{1}{\gamma}\right)}}. \tag{28}$$

**Lemma**

*(Single stock) Assume that $\hat{\mu} = \mu$ and $\varepsilon = \hat{\varepsilon}$. Then, the optimal strategy for a small benchmark portfolio can be found as*

$$\pi^* = \frac{1}{\varepsilon} - \left(\frac{\mu}{\lambda}\right)^{-\frac{1}{\gamma}} \left[\varepsilon\left(\frac{\mu}{\lambda\varepsilon}\right)^{\frac{\theta}{\eta}}\right]^{\left(\frac{1}{\gamma}-1\right)}. \tag{29}$$

# Benchmark portfolios:

- Market benchmark portfolio without individual jump risk:

  We obtain the optimal strategy of the agent for a jump-diffusion model under the same case in a market benchmark portfolio, plugging $\hat{\pi} = 1$ into the strategy (13).

**Lemma**

*(Single stock) Assume that $\hat{\mu} = \mu$, $\varepsilon = \hat{\varepsilon}$ and all agents are homogeneous. Then, the optimal strategy for a market benchmark portfolio is found as*

$$\pi^* = \frac{1}{\varepsilon} - \left(\frac{\mu}{\lambda}\right)^{-\frac{1}{\gamma}} \frac{(1-\varepsilon)^{\theta\left(1-\frac{1}{\gamma}\right)}}{\varepsilon^{\left(1-\frac{1}{\gamma}\right)}}. \tag{30}$$


(a) Optimal equilibrium strategy (left picture) and market benchmark portfolio strategy (right picture)


(b) Small benchmark portfolio strategy when $\eta = 1.1$ (left picture) and $\eta = 2$ (right picture), respectively

# Concluding Remarks

- We considered optimal portfolio investment strategies with CRRA utility function in a jump-diffusion market under the relative performance concerns.
- We found the results for one agent influenced by the group's decisions.
- We simplified the models in Lacker and Zariphopoulou [7] considering one agent and a group of agents and also recovered their results when $n \to \infty$ in their models for the pure-diffusion case.
- We found that the influence of the group affects the decision of the agent in combination with the degree of risk-aversion.
- We discovered that the number of the crashes in the market also affect the agent's strategy and even creates an arbitrage.

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*THANK YOU FOR YOUR ATTENTION!*