Model-free and data-driven methods in mathematical finance

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A paradigm shift in mathematical finance

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A paradigm shift in mathematical finance

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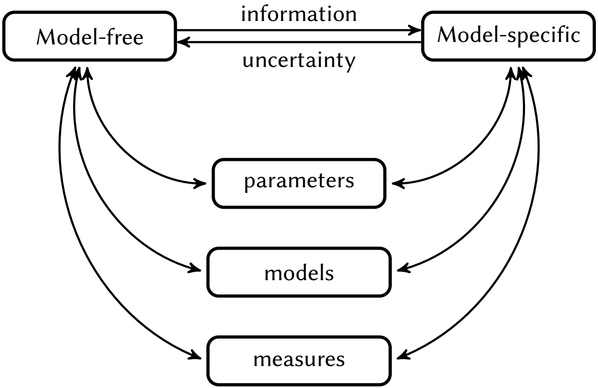
'New' paradigm:

You are not given the model and your task is to say something about option prices, value-at-risk, ... $\rightsquigarrow$ compute bounds

A paradigm shift in mathematical finance, II



Motivation

Coin tossing / Dice rolling

We are rolling two dices D_1, D_2 and are interested in the distribution of the sum.

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- Choices with dependent dices:
  - $D_1, D_2 = D_1$ (comonotonicity)
  - $D_1, D_2 = 7 - D_1$ (countermonotonicity)
  - $D_1, D_2 = D_1 + 1$ ("permutation")
  - ...

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Motivation

Random variables

- $(X_1, \dots, X_d)$: random variables with marginal distributions $(F_1, \dots, F_d)$
- Dependence structure: determined by joint distribution F or copula C
- Sklar's Theorem: given $F, F_1, \dots, F_d$, there exists C s.t.

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)) \quad \text{for all } x \in \mathbb{R}^d$$

- Dependence uncertainty: the marginal distributions are known, the dependence structure is not known

▶ Main question: f 'nice' function, compute

$$\inf\{\mathbb{E}_C[f] : C \text{ copula}\} \quad \text{and} \quad \sup\{\mathbb{E}_C[f] : C \text{ copula}\}$$

- Recently, the problem was reformulated under additional constraints by Tankov

$$\inf / \sup \{\mathbb{E}_C[f] : C \text{ copula } + \text{partial information on } C\}$$

- Math Finance: d'Aspremont, Bertsimas, Deelstra, Denuit, Hobson, Laurence, Vyncke, Wang, …
- QRM / Insurance Math: Bernard, Embrechts, Puccetti, Rüschendorf, Vanduffel, Wang, …

Outline

T. Lux

Improved Fréchet–Hoeffding bounds

T. Lux

Theorem

Let $S \subseteq \mathbb{I}^d$ be a compact set and Q^* be a d-quasi-copula. Consider the set

$$\mathcal{Q}^{S,Q^*} := \left\{Q \in \mathcal{Q}^d : Q(\mathbf{x}) = Q^*(\mathbf{x}) \text{ for all } \mathbf{x} \in S\right\}.$$

Then it holds that

$$\begin{aligned} \underline{Q}^{S,Q^*}(\mathbf{u}) \leq Q(\mathbf{u}) \leq \overline{Q}^{S,Q^*}(\mathbf{u}) \quad \text{for all } \mathbf{u} \in \mathbb{I}^d \\ \text{and} \quad \underline{Q}^{S,Q^*}(\mathbf{u}) = Q(\mathbf{u}) = \overline{Q}^{S,Q^*}(\mathbf{u}) \quad \text{for all } \mathbf{u} \in S \end{aligned} \tag{1}$$

for all $Q \in \mathcal{Q}^{S,Q^*}$, where the bounds $\underline{Q}^{S,Q^*}$ and $\overline{Q}^{S,Q^*}$ are provided by

$$\begin{aligned} \underline{Q}^{S,Q^*}(\mathbf{u}) &= \max\left(0, \sum_{i=1}^{d} u_i - d + 1, \max_{\mathbf{x}\in S}\left\{Q^*(\mathbf{x}) - \sum_{i=1}^{d}(x_i - u_i)^+\right\}\right) \\ \overline{Q}^{S,Q^*}(\mathbf{u}) &= \min\left(u_1, \ldots, u_d, \min_{\mathbf{x}\in S}\left\{Q^*(\mathbf{x}) + \sum_{i=1}^{d}(u_i - x_i)^+\right\}\right). \end{aligned} \tag{2}$$

Furthermore, the bounds $\underline{Q}^{S,Q^*}, \overline{Q}^{S,Q^*}$ are d-quasi-copulas.

Improved Fréchet–Hoeffding bounds, II

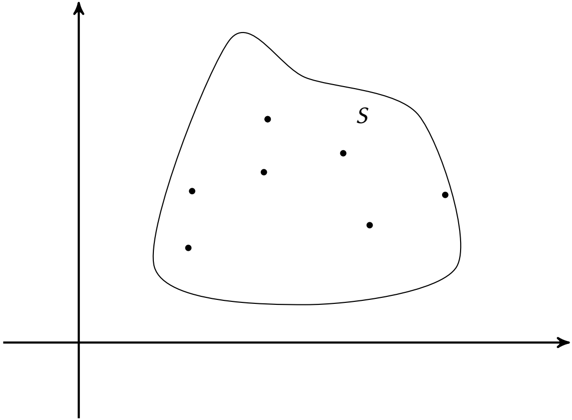

Figure: Illustration of the set S.

Questions — open problems

1. The 'nice' functions are Δ-tonic — basket options are excluded …
2. The improved Fréchet–Hoeffding bounds are not sharp for $d > 2$, although …
   - Tankov showed that they are copulas for $d = 2$,
   - Bernard et al. strengthened this result ($d = 2$).

   Are they pointwise sharp, e.g. $\overline{Q}(u) = \sup_{Q \in \mathcal{Q}_\star} Q(u)$?
3. The marginals are known. Is that realistic?

Outline

D. Bartl, M. Kupper, T. Lux, S. Eckstein

Transport and relaxed transport duality

Aim: upper bound – superhedging strategy for $f(\mathbf{X}) \rightsquigarrow \mathbb{E}[f(\mathbf{X})]$

Classical ingredients:

- $\psi_1, \ldots, \psi_d : \mathbb{R} \to \mathbb{R}$ bounded, measurable functions ('put options')
- $\nu_1, \ldots, \nu_d$ marginal distributions, μ joint distribution

Then

$$\sup_{\mu \in \ldots} \int f \mathrm{d}\mu = \inf \left\{ \int \psi_1 \mathrm{d}\nu_1 + \cdots + \int \psi_d \mathrm{d}\nu_d : \psi_1 + \cdots + \psi_d \geq f \right\}$$

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New ingredients

- π^i price of multi-asset digital 1_{A^i}, $A^i = \times_{j=1}^{d}(-\infty, A^i_j]$, $i \in I$
- a^i amount invested in 1_{A^i}

Transport and relaxed transport duality, II

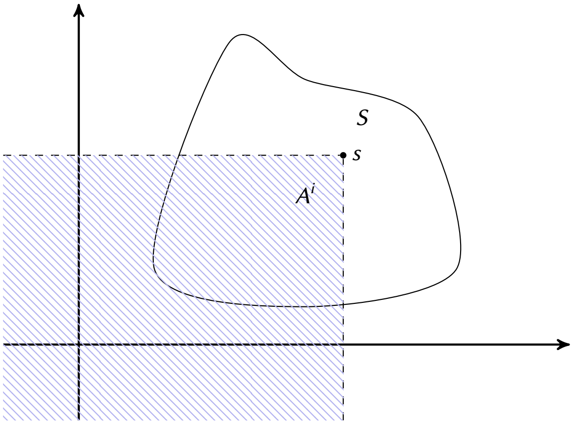

Figure: Illustration of the relation between the sets S and $(A^i)_{i \in I}$.

Questions — open problems

1. The additional information is not stemming from traded assets, *i.e.* multi-asset digital options are not (liquidly) traded …
2. Can we replace the additional information with traded asset prices, *e.g.* basket options?

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Outline

E. Dragazi, S. Liu

Transport duality under option-implied information

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New ingredients

- p_i price of multi-asset option with payoff ϕ_i
- b_i amount invested in ϕ_i

Transport duality under option-implied information, II

“Primal” side: consistent measures

$$\mathcal{Q} = \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \mu_j = \nu_j, j \in \mathcal{J}, \text{ and } \int_{\mathbb{R}^d} \phi_i \mathrm{d}\mu = p_i, i \in \mathcal{I} \right\}$$

“Dual” side: trading strategies and cost

$$\Theta(f) = \left\{ (\psi, b) : \sum_{j \in \mathcal{J}} \psi_j + \sum_{i \in \mathcal{I}} b_i \phi_i \geq f \right\}$$

$$\pi(\psi, b) = \sum_{j \in \mathcal{J}} \int_{\mathbb{R}} \psi_j \mathrm{d}\nu_j + \sum_{i \in \mathcal{I}} b_i p_i$$

Definition

A trading strategy (ψ, b) that satisfies

$$(\psi, b) \in \Theta(\epsilon) \quad \text{and} \quad \pi(\psi, b) = \sum_{j \in \mathcal{J}} \int_{\mathbb{R}} \psi_j \mathrm{d}\nu_j + \sum_{i \in \mathcal{I}} b_i p_i \leq 0$$

for some $\epsilon > 0$, is called *uniform strong arbitrage.*

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Transport duality under option-implied information, III

Theorem (Fundamental Theorem)

There does not exist a uniform strong arbitrage strategy in the market if and only if the set $\mathcal{Q}$ is non empty.

Theorem (Superhedging Duality)

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous and bounded function. Assuming there does not exist uniform strong arbitrage in the market, then

$$\max_{\mu \in \mathcal{Q}} \int_{\mathbb{R}^d} f \, \mathrm{d}\mu = \inf \left\{ \pi(\psi, b) : (\psi, b) \in \Theta(f) \right\}$$
$$=: \Phi(f)$$

Numerical scheme – penalization and neural networks

Eckstein & Kupper (AMO, 2019)

We would like to approximate the function

$$\Phi(f) = \inf\left\{ \sum_j \int \psi_j \mathrm{d}\nu_j + \sum_i b_i p_i \,\middle|\, \sum \psi_j + \sum b_i \phi_i \geq f \right\}$$

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Step 1: Neural network approximation

$$\Phi^m(f) = \inf_{\psi_j \in \mathcal{H}^m} \left\{ \sum_{j \in \mathcal{J}} \int \psi_j \mathrm{d}\nu_j + \sum_{i \in \mathcal{I}} b_i p_i : \sum_{j \in \mathcal{J}} \psi_j + \sum_{i \in \mathcal{I}} b_i \phi_i \geq f \right\}.$$

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Step 2: Penalization

$$\Phi^m_{\theta,\gamma}(f) = \inf_{\psi_j \in \mathcal{H}^m} \left\{ \sum_{j \in \mathcal{J}} \int \psi_j \mathrm{d}\nu_j + \sum_{i \in \mathcal{I}} b_i p_i + \int \beta_\gamma \left(f - \sum_{j \in \mathcal{J}} \psi_j - \sum_{i \in \mathcal{I}} b_i \phi_i \right) \mathrm{d}\theta \right\}.$$

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Numerical results, I

Convergence

- 3 assets; Black–Scholes dynamics with Gaussian copula
- Additional information (ϕ) and payoff (f): call-on-max, *i.e.*

$$\left(\max(S^1, S^2, S^3) - K, 0 \right)^+$$

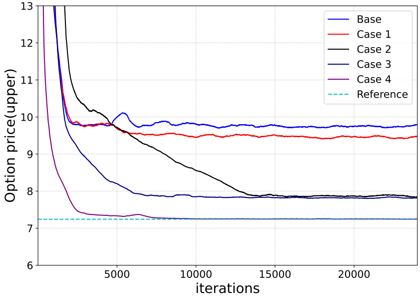

Figure: Bounds on option prices

Numerical results, II

3 assets; impact of additional information

0. Base case: only the three marginal distributions are known.
1. Case 1: Base plus additional traded two-asset call-on-max options with payoff function
$$\phi_1(x) = \left(x_1 \vee x_2 - K\right)^+, \quad K = 6.$$
2. Case 2: Case 1 plus additional traded two-asset call-on-max options with payoff function
$$\phi_2(x) = \left(x_2 \vee x_3 - K\right)^+, \quad K = 6.$$
3. Case 3: Case 2 plus additional traded two-asset call-on-max options with payoff function
$$\phi_3(x) = \left(x_1 \vee x_3 - K\right)^+, \quad K = \{5, 6, 7\}.$$
4. Case 4: Case 3 plus additional traded three-asset call-on-max options with payoff function
$$\phi_4(x) = \left(x_1 \vee x_2 \vee x_3 - K\right)^+, \quad K = \{5, 7\}.$$

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Numerical results, II (cont'd)

3 assets; impact of additional information

5. Case 5: Base case plus 5 more options with payoff ϕ_1 and strike prices $K = \{6, 9, 11, 13, 15\}$.
6. Case 6: Case 5 plus 4 more options with payoff ϕ_2 and strike prices $K = \{6, 11, 13, 15\}$.
7. Case 7: Case 6 plus 5 more options with payoff ϕ_3 and strike prices $K = \{5, 6, 7, 11, 13\}$.
8. Case 8: Case 7 plus 2 more options with payoff ϕ_4 and strike prices $K = \{5, 7\}$.

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Numerical results, II (cont'd)

3 assets; impact of additional information

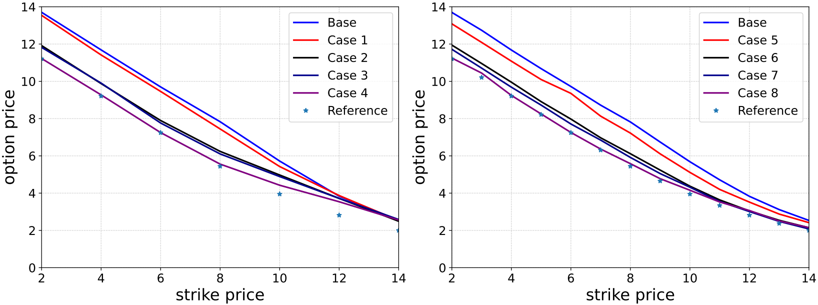

Figure: Model-free bounds for various strikes using Cases 1–4 on the left, and Cases 5–8 on the right.

Numerical results, III

6 assets; impact of additional information

0. Base case: only the six marginal distributions are known.
1. Case 1: Base plus call-on-min options with payoff $(x_1 \wedge \cdots \wedge x_6 - K)^+$ for 8 strike prices
$$K = \{6.5, 7.5, 8.5, 9.5, 10.5, 11.5, 12.5, 13.5\}.$$
2. Case 2: Case 1 plus call-on-max options with payoff $(x_1 \vee \cdots \vee x_6 - K)^+$ for 8 strike prices
$$K = \{6.5, 7.5, 8.5, 9.5, 10.5, 11.5, 12.5, 13.5\}.$$
3. Case 3: Case 2 plus basket options with payoff
$$\left(\frac{1}{5}\sum_i x_i - K\right)^+ \text{ for } i \in \{1, \ldots, 5\},\ i \in \{2, \ldots, 6\},\ i \in \{1, 2, 3, 5, 6\},$$
each with 8 strike prices $K = \{6.6, 7.6, 8.6, 9.6, 10.6, 11.6, 12.6, 13.6\}$.

Numerical results, III (cont'd)

6 assets; impact of additional information

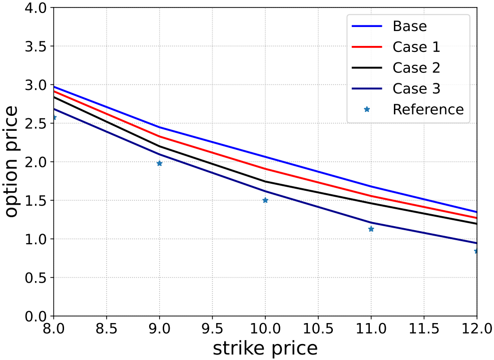

Figure: Model-free bounds for various strikes using the setting 0−3.

Numerical results, III (cont'd)

6 assets; impact of additional information

4. Case 4: Base plus three put-on-min options with payoff $(K - x_i \wedge x_j)^+$ for $\{i,j\} \in \{\{1,2\},\{3,4\},\{5,6\}\}$, each with 8 strike prices
$$K = \{6.75, 7.75, 8.75, 9.75, 10.75, 11.75, 12.75, 13.75\}.$$
5. Case 5: Case 4 plus call-on-min options with payoff $(x_1 \wedge \cdots \wedge x_6 - K)^+$ for 8 strike prices
$$K = \{6.5, 7.5, 8.5, 9.5, 10.5, 11.5, 12.5, 13.5\}.$$
6. Case 6: Case 5 plus call-on-max options with payoff $(x_1 \vee \cdots \vee x_6 - K)^+$ for 8 strike prices
$$K = \{6.5, 7.5, 8.5, 9.5, 10.5, 11.5, 12.5, 13.5\}.$$
7. Case 7: Case 6 plus basket options with payoff
$$\left(\frac{1}{5}\sum_i x_i - K\right)^+ \text{ for } i \in \{1,\dots,5\},\ i \in \{2,\dots,6\},\ i \in \{1,2,3,5,6\},$$
each with 8 strike prices $K = \{6.6, 7.6, 8.6, 9.6, 10.6, 11.6, 12.6, 13.6\}$.
8. Case 8: Base plus basket options with payoff
$$\left(\frac{1}{5}\sum_i x_i - K\right)^+ \text{ for } i \in \{1,\dots,5\},\ i \in \{2,\dots,6\},\ i \in \{1,2,3,5,6\},$$
each with 8 strike prices $K = \{6.6, 7.6, 8.6, 9.6, 10.6, 11.6, 12.6, 13.6\}$.

Numerical results, III (cont'd)

6 assets; impact of relevant information

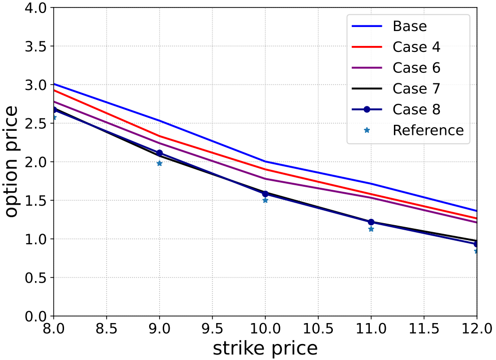

Figure: Model-free bounds for various strikes using the setting 0, 4–8.

Computational times

# Assets	Time (sec)
6	498
15	921
18	1107

Table: Dimension vs computational time.

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THANK YOU